TRIGONOMETRIC SERIES WITH QUASI-MONOTONE COEFFICIENTS

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1. **Introduction.** In this note we extend the following Theorem A of Chaundy and Jolliffe [3] and Theorems B and C of Boas [2].

THEOREM A. *Suppose that* $b_n \downarrow 0$. *A necessary and sufficient condition that the series*

$$(1.1) \qquad S(x) = \sum_{1}^{\infty} b_n \sin nx$$

should be uniformly convergent is that $nb_n = o(1)$.

THEOREM B. *If* $b_n \downarrow 0$ *and* $0 \leqq \gamma \leqq 1$, *then*

$$(1.2) \qquad x^{-\gamma} S(x) \in L(0, \pi)$$

if and only if the series

$$(1.3) \qquad \sum_{1}^{\infty} n^{\gamma-1} b_n$$

converges.

THEOREM C. *If* $b_n \downarrow 0$ *and* $0 < \gamma < 1$ *and*

$$(1.4) \qquad C(x) = \sum_{1}^{\infty} b_n \cos nx,$$

then $x^{-\gamma} C(x) \in L(0, \pi)$ *if and only if* (1.3) *converges.*

G. Sunouchi proved Theorems B and C by a different method [7] and Aljančić, Bojanić and Tomić [1], and S. O'Shea [6] have extended these theorems in different directions.

2. **Statement of results.** A sequence (b_n) of nonnegative numbers is said to be quasi-monotone if

$$(2.1) \qquad b_{n+1} \leqq b_n(1 + \alpha/n)$$

for some constant $\alpha > 0$ and all $n > n_0(\alpha)$ [5; 8]. An equivalent definition is that $b_n/n^{\beta} \downarrow 0$ for some $\beta > 0$. We may suppose that α is an integer. Let $P(n)$ denote the number of terms b_k such that $k \leqq n$,

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$b_{k-1} < b_k$; and let $\phi(x)$ be any function, positive and nondecreasing for $x \geqq 1$, and such that

(i) $$\phi(x^2) \leqq c\phi(x)$$

for $x > 1$ and some positive constant c and

(ii) $$\sum_1^\infty 1/n\phi(n) < \infty.$$

THEOREM 1. *Let* (b_n) *be quasi-monotone.*

(a) *If either*

(2.2) $$b_n = o(1), \qquad P(n) = O(n/\phi(n)),$$

or

(2.3) $$\sum_1^\infty n^{-1} b_n$$

is convergent, then

(2.4) $$\sum_1^\infty | b_n - b_{n+1} |$$

is convergent.

(b) *If* $nb_n = o(1)$, *then*

$$\sum_p^\infty | b_n - b_{n+1} | \leqq \frac{(4\alpha + 1)}{p} \max_{k \geqq p} (kb_k).$$

(c) *If either*

(2.5) $$P(n) = O(n/\phi(n) \log n)$$

and (2.3) *is convergent, or*

(2.6) $$\sum_1^\infty (\log n/n) b_n$$

is convergent, then

(2.7) $$\sum_1^\infty | b_n - b_{n+1} | \log n$$

is convergent.

(d) *If* $0 < \gamma \leqq 1$ *and* (1.3) *is convergent, then*

(2.8) $$\sum_1^\infty | b_n - b_{n+1} | n^\gamma$$

is convergent.

THEOREM 2. *Let* (b_n) *be quasi-monotone. If either* (2.2) *holds or* (1.3) *is convergent for* $\gamma=0$, *then* (1.1) *and* (1.4) *are convergent for every* x, *save possibly* $x=0$ *in the case of* (1.4).

THEOREM 3. *Let* (b_n) *be quasi-monotone. A necessary and sufficient condition that the series* (1.1) *should be uniformly convergent throughout any interval is that* $nb_n=o(1)$.

THEOREM 4. *Let* (b_n) *be quasi-monotone.*

(i) *If* $0<\gamma<1$, *and* (1.3) *is convergent, then* $x^{-\gamma}C(x)\in L(0, \pi)$.

(ii)[1] *If* $0<\gamma\leqq 1$, *and* (1.3) *is convergent, then* $x^{-\gamma}S(x)\in L(0, \pi)$.

(iii) *If* (1.3) *is convergent for* $\gamma=0$, *and* (2.5) *holds, then*

$$C(x) \quad and \quad S(x) \in L(0, \pi).$$

REMARKS. I. Let $b_n\geqq 0$ (not necessarily monotone). Then [**2**, pp. 219–220] $x^{-\gamma}S(x)\in L(0, \pi)$, $0<\gamma\leqq 1$ implies the convergence of[1] (1.3) and $x^{-\gamma}C(x)\in L(0, \pi)$, $0<\gamma<1$ implies the convergence of (1.3).

II. If $S(x)\in L(0, \pi)$ is odd and (1.1) its Fourier series, then $\sum b_n/n<\infty$ [**4**, p. 30; **9**, p. 59]. Hence, if (b_n) be quasi-monotone, then (2.4) is convergent, and (1.1) is convergent everywhere and uniformly convergent over the interval $0<\delta\leqq x\leqq 2\pi-\delta$.

These remarks along with Theorem 4 give extensions of Theorems B and C for series with quasi-monotone coefficients. We show by an example that the condition (2.2) on $P(n)$ is best possible in the sense that if $\sum 1/n\phi(n)=\infty$, then (2.4) may not converge.

3. **Proof of Theorem 1.** (a) Write $b_n=b(n)$ and let

$$S(p, q) = \sum_{p}^{q} \left| b(n) - b(n+1) \right| .$$

If $P(k)=O(1)$, then the convergence of (2.4) follows easily and so we assume $\lim_{k\to\infty} P(k)=\infty$. Let $n_1\geqq\max(n_0, 2)$ be the least integer such that $b(n_1+1)>b(n_1)$; $p_1\geqq 1$ the largest integer such that $b(n_1+t)>b(n_1+t-1)$ for $t=1, 2, \cdots, p_1$; $n_2>n_1+p_1$ the least integer such that $b(n_2+1)>b(n_2)$ and so on. Then $n_k\uparrow\infty$. Let $n_k\leqq p<n_{k+1}$, $n_t\leqq q<n_{t+1}$. Then $S(n_j, \ n_{j+1}-1)=\sum_1 (b(n+1)-b(n))+\sum_2 (b(n)-b(n+1))$ where the summation in $\sum_1$ is given by $n_j\leqq n\leqq n_j+p_j-1$ and in $\sum_2$ by $n_j+p_j\leqq n\leqq n_{j+1}-1$. Hence

$$S(p, q) \leqq 2 \sum_{k}^{t} \{b(n_j + p_j) - b(n_j)\} + b(n_k). \tag{3.1}$$

[1] Compare [**6**, p. 281].

Denote by $A, A_1, \cdots$ positive constants which may depend on α and c. Then from (2.2)

$$p_1 + \cdots + p_k \leqq P(n_k + p_k) < A(n_k + p_k)/\phi(n_k + p_k).$$

Hence for all large k, $p_k \leqq n_k$ and so for $k = 1, 2, \cdots$, $P(n_k + p_k) \equiv P_k < A_1 n_k/\phi(n_k)$. Write $n_j = n$, $p_j = p$. Then from (2.1)

$$b(n + p)/b(n) \leqq \Pi(p, n, \alpha)$$

where

$$\begin{aligned}\Pi(p, n, \alpha) &= (1 + \alpha/n) \cdots (1 + \alpha/(n + p - 1)) \\ &< (1 + p/n)^\alpha \leqq 1 + \alpha\cdot 2^{\alpha-1}(p/n).\end{aligned}$$

Hence from (3.1), we have

$$S(p, q) < \left(\max_{j \geqq k} b(n_j)\right)\left(1 + A_2 \sum_{j=k}^{t} p_j/n_j\right). \tag{3.2}$$

Now

$$\sum_{j=k}^{t} p_j/n_j < A_1\left[\sum_{k}^{t} \frac{1}{\phi(n_j)}(1 - n_j/n_{j+1}) + \frac{1}{\phi(n_k)} + \frac{1}{\phi(n_{t+1})}\right]. \tag{3.3}$$

Since

$$\sum_{n_j}^{n_{j+1}-1} 1/n\phi(n) > \frac{1}{\phi(n_{j+1})}(1 - n_j/n_{j+1}),$$

$$\sum_{1}^{\infty} \frac{1}{\phi(n_{j+1})}(1 - n_j/n_{j+1}) < \infty;$$

and hence

$$\sum_{1}^{\infty} \frac{1}{\phi(n_j)}(1 - n_j/n_{j+1}) \tag{3.4}$$

is convergent, provided $n_{j+1} \leqq n_j^2$ for all large j. If $n_{j+1} > n_j^2$ for an infinity of j, then we define n_j^* as follows:

$$n_1^* = n_1, \qquad n_j^* = \min(n_j, n_{j-1}^{*2}), \qquad j > 1. \tag{3.5}$$

It is easily seen that $n_{j-1}^* < n_j^* \leqq n_{j-1}^{*2}$ for $j \geqq 2$ and

$$\frac{1}{\phi(n_j)}(1 - n_j/n_{j+1}) < \frac{A_3}{\phi(n_j^*)}(1 - n_j^*/n_{j+1}^*).$$

By the above argument

$$\sum \{1/\phi(n_j^*)\}(1 - n_j^*/n_{j+1}^*) < \infty,$$

and hence (3.4) is convergent. From (3.2), (3.3) and the convergence of (3.4), the convergence of (2.4) follows. If (2.3) is convergent, then since $\{b(n)/n\}$ is quasi-monotone [5], $b(n) = o(1)$. Further, from (2.1),

$$b(n + p) - b(n) \leqq \alpha\left\{\frac{b(n)}{n} + \cdots + \frac{b(n + p - 1)}{n + p - 1}\right\}. \tag{3.6}$$

Hence from (3.1), the convergence of (2.4) follows.

(b) If $n_k \leqq p < n_k + p_k$, then

$$S(p, n_{k+1} - 1) = 2b(n_k + p_k) - b(p) - b(n_{k+1}),$$

and if $n_k + p_k \leqq p \leqq n_{k+1} - 1$, then

$$S(p, n_{k+1} - 1) = b(p) - b(n_{k+1}).$$

Let $\mu(n) = \max_{k \geq n} (kb_k)$. Then

$$S(n_{k+1}, n_{t+1} - 1) = \sum_{j=k+1}^{t} \{2b(n_j + p_j) - b(n_j) - b(n_{j+1})\}$$
$$\leqq 2\alpha \sum_{n_{k+1}}^{\infty} \{b(n)/n\} + b(n_{k+1}).$$

Since

$$\sum_{m}^{\infty} b(n)/n \leqq 2\mu(m)/m, \qquad S(p, \infty) \leqq (4\alpha + 1)\mu(p)/p.$$

(c) Let

$$S(p, q) = \sum_{p}^{q} | b_n - b_{n+1} | \log n$$

where as in part (a), $n_k \leqq p < n_{k+1}$, $n_t \leqq q < n_{t+1}$. Then

$$\begin{aligned} S(p, q) < \sum_{2}^{n_t+1} b(n)/(n - 1) \\ + \sum_{j=1}^{t} \{b(n_j + p_j) \log (n_j + p_j) - b(n_j) \log n_j \\ + b(n_j + p_j) \log (n_j + p_j - 1) - b(n_{j+1}) \log (n_{j+1} - 1)\}. \end{aligned} \tag{3.7}$$

Now since $\sum b(n)/n$ is convergent, $b(n) = o(1)$ [5] and so

$$S(p, q) < O(1) + A_4\left[\sum_{j=1}^{t} p_j \log n_j/n_j + A_5\right]. \tag{3.8}$$

From (2.5) we have

$$\tag{3.9} \begin{aligned} &\sum_{j=1}^{t} (p_j \log n_j)/n_j \\ &< A_6\left[\sum_{j=1}^{t} \frac{1}{\phi(n_j)}\left(1 - \frac{\log n_{j+1}}{n_{j+1}} \Big/ \frac{\log n_j}{n_j}\right) + \frac{n_t \log n_{t+1}}{\phi(n_t) n_{t+1} \log n_t}\right]. \end{aligned}$$

If $n_{j+1} \leqq n_j^2$ for all large j, then the series

$$\tag{3.10} \sum_{1}^{\infty} \frac{1}{\phi(n_j)}\left\{1 - \frac{\log n_{j+1}}{n_{j+1}} \Big/ \frac{\log n_j}{n_j}\right\}$$

is convergent. If $n_{j+1} > n_j^2$ for an infinity of j, we define n_j^*, as in (3.5), and then prove the convergence of (3.10). From (3.8), (3.9) and the convergence of (3.10), the convergence of (2.7) follows.

To prove the second part of (c), we observe that for $\theta = 0, 1$

$$\begin{aligned} &b(n_j + p_j) \log (n_j + p_j - \theta) - b(n_j) \log (n_j - \theta) \\ &\qquad < A_7\left[\sum_{t=0}^{p_j-1} \frac{b(n_j + t) \log (n_j + t)}{n_j + t}\right]. \end{aligned}$$

Hence from (3.7) and the convergence of (2.6), the convergence of (2.7) follows.

(d) We omit the proof of the convergence of (2.8) which is similar to the one used to prove the second part of (c).

4. **Proof of Theorem 2.** This follows from the convergence of (2.4). We note that both series (1.1) and (1.4) are uniformly convergent over the interval $0 < \delta \leqq x \leqq 2\pi - \delta$.

Proof of Theorem 3. We need consider the interval $0 \leqq x \leqq \pi$. To prove that the condition is necessary, consider

$$S(p, q, x) = \sum_{p}^{q} b(n) \sin nx,$$

and let $p = [(1/2)q + 1]$, $x = \pi/2q$. Then $b(r) \geqq b(q)/\Pi$ where $p \leqq r < q$ and

$$\Pi = \left(1 + \frac{\alpha}{q - p}\right) \cdots \left(1 + \frac{\alpha}{q - 1}\right) < A_8, \qquad q > 3.$$

Hence

$$\begin{aligned} S(p, q, x) &\geq b(q) A_8^{-1}\{\sin px + \cdots + \sin qx\} \\ &> A_8^{-1} b(q) \left(\frac{q}{2} - 1\right) \sin \left(\frac{\pi}{4}\right). \end{aligned}$$

Since the series is uniformly convergent, $qb(q)=o(1)$. We now show that the condition is sufficient. Let $nb(n)<\epsilon$ for $n\geqq N(\epsilon)$ and let $\max(N, n_1)\leqq p<q$. If $x\leqq\pi/q$, then $|S|\leqq pxb(p)+\cdots+qxb(q)<xq\epsilon\leqq\pi\epsilon$. If $x\geqq\pi/p$, then from Theorem 1(b)

$$|S| \leqq \operatorname{cosec}(x/2)\left\{\sum_{p}^{q}|b(n)-b(n+1)| + b(p)+b(q+1)\right\}$$

$$\leqq p\left\{\frac{\mu(p)(4\alpha+1)}{p}+\frac{2\mu(p)}{p}\right\} \leqq (4\alpha+3)\epsilon.$$

If $\pi/p<x<\pi/q$ then, with $k=[\pi/x]$, we have

$$|S| \leqq |S(p,k)| + |S(k+1,q)|$$

$$\leqq xk\epsilon + \frac{\pi}{x}\left\{\frac{(4\alpha+3)\mu(k+1)}{k+1}\right\} \leqq \epsilon\{\pi+4\alpha+3\}$$

and the theorem is proved.

Proof of Theorem 4. Boas [2] proved that if $b_n=o(1)$ and

$$\sum |b(n-1)-b(n+1)|\, n^{\gamma} \tag{4.1}$$

is convergent, then $x^{-\gamma}S(x)\in L(0,\pi)$ where $0<\gamma\leqq 1$, and $x^{-\gamma}C(x)\in L(0,\pi)$ where $0<\gamma<1$. By Theorem 1(d), the series (2.8) and hence (4.1) are convergent. Further from the convergence of (1.3) we have $b_n=o(1)$ [5]. Hence (i) and (ii) are proved. To prove (iii) we note that

$$\int_0^{\pi}\frac{1-\cos nx}{x}dx < A_9\log n, \qquad \int_0^{\pi}\frac{|\sin nx|}{x}dx < A_9\log n.$$

Hence by the argument of Boas [2], $C(x)$ and $S(x)\in L(0,\pi)$ if

$$\sum |b(n-1)-b(n+1)| \log n \tag{4.2}$$

is convergent. From Theorem 1(c), (2.7) and hence (4.2) are convergent and (iii) is proved.

5. **Example.** *Let $\phi(x)$ be positive and nondecreasing for $x\geqq 1$, $\phi(x^2)\leqq c\phi(x)$ for $x>1$ and $\sum 1/n\phi(n)=\infty$. Then there exists a quasi-monotone sequence (b_n) for which*

$$b_n=o(1),\ P(n)=O(n/\phi(n)) \quad \text{and} \quad \sum_1^{\infty}|b_n-b_{n+1}|=\infty. \tag{5.1}$$

Let $p_i=1+i$, $q_i=[i\phi(i)]+1$, $i=1,2,\cdots,$

$$n_1=2, \qquad n_{i+1}=n_i+p_i+q_i, \qquad i=1,2,\cdots,$$

$$S(n)=\sum_1^{n}1/k\phi(k),$$

$$b_n \equiv b(n) = 1/S(n_1) \qquad \text{for } n = 1,$$
$$b(n_k) = 1/S(n_k) \qquad \text{for } k = 1, 2, \cdots,$$
$$b(n_k + t) = b(n_k)(n_k + t)/n_k \qquad \text{for } t = 1, 2, \cdots p_k;\ k = 1, 2, \cdots,$$
$$b(n_k + p_k + t) = b(n_{k+1}) \text{ for } t = 1, \cdots, n_{k+1} - n_k - p_k - 1;\ k = 1, 2, \cdots.$$

Then (b_n) is quasi-monotone and conditions (5.1) are satisfied.

Added in proof. (i) In Theorem 1 (a), (c), it is not necessary to suppose that $\phi(x^2) \leqq c\phi(x)$. In fact it can be proved that *if* $\lambda_n > 0$, $\uparrow$ *for* $n = 1, 2, \cdots$, $\phi(x) > 0$, $\uparrow$ *for* $x \geqq \lambda_1$, $\int_{\lambda_1}^{\infty} dx/\{x\phi(x)\} < \infty$, *then* $\sum_1^{\infty} \{\lambda_n/\phi(\lambda_n)\}(\lambda_n^{-1} - \lambda_{n+1}^{-1}) < \infty$.

(ii) The following theorem can be proved by the argument of P. Szüsz (Acta Math. Acad. Sci. Hungar. **12** (1961), 215–220). *Let K be any arbitrary large positive number and suppose there exists a sequence (n_k), $k = 1, 2, \cdots$ of natural numbers with the following properties*: (a) $n_{k+1}/n_k > K$; $k = 1, 2, \cdots$, (b) $\sum_1^{\infty} a_k |\sin \pi n_k x| < \infty$ *for some* $x \neq 0, \neq 1, \cdots$, *where (a_n) is quasi-monotone. Then* $\sum_1^{\infty} a_k < \infty$.

References

1. S. Aljančić, R. Bojanić et M. Tomić, *Sur l'intégrabilité de certaines séries trigonométriques*, Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 67–84.
2. R. P. Boas, Jr., *Integrability of trigonometric series.* III, Quart. J. Math. Oxford Ser. (2), **3** (1952), 217–221.
3. T. W. Chaundy and A. E. Jolliffe, *The uniform convergence of a certain class of trigonometric series*, Proc. London Math. Soc. (2), **15** (1916), 214–216.
4. G. H. Hardy and W. W. Rogosinski, *Fourier series*, Cambridge Univ. Press, Cambridge, 1944.
5. S. M. Shah, *A note on quasi-monotone series*, Math. Student **15** (1947), 19–24.
6. S. O'Shea, *Note on an integrability theorem for sine series*, Quart. J. Math. Oxford Ser. (2), **8** (1957), 279–281.
7. G. Sunouchi, *Integrability of trigonometric series*, J. Math. Tokyo **1** (1953), 99–103.
8. O. Szász, *Quasi-monotone series*, Amer. J. Math. **70** (1948), 203–206.
9. A. Zygmund, *Trigonometrical series*, Vol. 1, Cambridge Univ. Press, New York, 1959.

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