CONTINUITY IN TOPOLOGICAL GROUPS[1]

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1. In the theory of topological groups, it is customary to make certain assumptions concerning the continuity of the product and continuity of the inverse. It has been noted that certain types of group spaces with less stringent assumptions than those usually made yield the ordinary assumptions [**1**; **2**; **3**; **4**; **5**]. In this note, we shall give a type of group space which is more general than the ones given in [**1**; **2**; **5**]. In fact, we will prove the following theorem:

THEOREM. *Suppose G is a second countable and second category regular space. If G is a group and its group multiplication is continuous separately, then G is a topological group under the group operation.*

The tool which we shall use is the Fort theorem [**6**]. We have pleasure in thanking Professor P. S. Mostert for his encouragement and helpful comments.

2. Let T and T^* be two topologies for a set Y. We say T is categorically related to T^* if and only if for every topological space X and function f on X into Y, $f-T^*$-continuous at each point of X implies that f is T-continuous at each point of a residual subset of X. Let T and T^* be topologies for a set Y, we say that the ordered pair (T, T^*) satisfies condition (a) if and only if there exists a sequence $U_1, U_2, \cdots$; and $K_1, K_2, \cdots$ subsets of Y such that (i) $U_n \subset K_n$ for each n; (ii) if $p \in U \in T$, then there exists n such that $p \in U_n \subset K_n \subset U$; (iii) if $q \in U_n$ then there exists $V \in T^*$ such that $q \in V$ and $V - K_n \in T^*$.

FORT THEOREM [**6**]. *If T and T^* are topologies for Y and (T, T^*) satisfies condition* (a), *then T is categorically related to T^*.*

Let $F_i \subset X$, $U_i \subset Y$, then $[F_1, F_2, \cdots, F_n: U_1, U_2, \cdots, U_n] = [f \in Y^X | f(F_i) \subset U_i \text{ for } i=1, \cdots, n]$. Let $C[G, G]$ denote the set of all continuous functions from G to G.

3. PROOF OF THE THEOREM. It is clear that we can embed the group G into $C[G, G]$ and in fact the topology of G is equivalent to the point-open topology of $C[G, G]$; we shall denote this topology by (G, T^*).

Let $[W_i | W_i \text{ closed and int } W_i \neq \square]$ be a countable fundamental neighborhood system of G, $[V_i | V_i \text{ open}]$ be a countable fundamental neighborhood system of G. Let $\mathfrak{F} = [F | F = \text{some } CV_i \text{ or some } W_i]$,

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$\mathfrak{R} = [R \mid R = \text{some } CW_i \text{ or some } V_i]$. Then $\mathfrak{R}$ or $\mathfrak{F}$ is countable. We shall topologize $G \subset C[G, G]$ by using

$$\mathfrak{J} = \{ [F_1, \cdots, F_n : R_1, \cdots, R_n] \mid F_i \in \mathfrak{F}, R_i \in \mathfrak{R} \}$$

as base for the open sets. Let (G, T) denote the topological space of G topologized in this way. Since $\mathfrak{J}$ is countable, we can arrange it into a sequence. Let $U_n = [F_{n_1}, \cdots, F_{n_p} : R_{n_1}, \cdots, R_{n_p}]$ be the nth element in this arrangement. Let $K_n = [F_{n_1}, \cdots, F_{n_p} : \overline{R}_{n_1}, \cdots, \overline{R}_{n_p}]$. Then it is clear that K_n is closed in point open topology (G, T^*). It is clear that (T, T^*) satisfies condition (a). By the Fort Theorem, (T, T^*) is categorically related.

Define the identity map i from G to G (i.e., $i(g) = g$ for $g \in G$). Since $i: (G, T^*) \to (G, T^*)$ is continuous, $i: (G, T^*) \to (G, T)$ is continuous at each point of some residual set of (G, T^*). But (G, T^*) is second category, so its residual set is not empty. By a standard argument, we know that (G, T) is equivalent to (G, T^*).

We have to prove the inversion of (G, T) is continuous, i.e. the map $\phi: (G, T) \to (G, T)$ is continuous, where $\phi(g) = g^{-1}$, g and g^{-1} belong to $(G, T) \subset C[G, G]$. We note that g and g^{-1} are one to one onto maps from G to G, hence if $g \in [F, R] \in \mathfrak{J}$, we can assume without loss of generality, that F and G are not equal to G, then since $g(F) \subset R$ and g is one to one onto, we have $g(G-F) = G - g(F) \supset G - R$ (i.e. $g^{-1}(G-R) \subset G-F$, $g^{-1} \in [G-R, G-F]$). Thus we have $g \in [F, R] \in \mathfrak{J}$ if and only if $g^{-1} \in [G-R, G-F] \in \mathfrak{J}$. Because $\{ [F, R] \mid F \in \mathfrak{F}, R \in \mathfrak{R} \}$ is the subbase for the space (G, T), ϕ is continuous.

If we define the function θ:

$$\theta: G \times G \to G$$

such that $\theta(x, y) = xy$, then $\theta: (G, T) \times (G, T^*) \to (G, T^*)$ is continuous. Hence $\theta: (G, T^*) \times (G, T^*) \to (G, T^*)$ is continuous, because (G, T) and (G, T^*) are equivalent. Hence we know G is a topological group.

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