so

\[(7) \quad (y_2y'_1 - y_1y'_2)(x) = \int_{a}^{x} y_2(s)y_1(s)[F(y_2(s), s) - F(y_1, s)]\, ds.\]

By hypothesis \((y_2y'_1 - y_1y'_2)(b) = 0\). The integrand in (7) is positive, however, as long as \(y_2(s) > y_1(s)\) and in particular on some interval \((a, a)\) — because \((y_2 - y_1)'(a) > 0\). In fact, \(a\) may be taken as \(b\) because the same argument as in Lemma 1 shows that the graphs of \(y_2(x)\) and \(y_1(x)\) can not intersect on \((a, b)\). Thus the right side of (7) does not tend to zero as \(x \to b^-\).

**Proof of the theorem.** The existence of at least one solution of (1) + (C) has been proved by Nehari [1, Theorem IV]. By the preceding lemmas there is at most one such solution.

**Reference**


**Massachusetts Institute of Technology**

---

**ON THE MEASURABILITY OF FUNCTIONS IN TWO VARIABLES**

MARK MAHOWALD

Let \((X, \mu)\) and \((Y, \nu)\) be two compact spaces having regular Borel measures defined on them. By a measurable modification \(f(x, y)\) of a function \(f(x, y)\) we mean a function measurable in both variables together and for which \(f(x- y) = f(x)\) almost everywhere \([\nu]\) for every \(x\).

The purpose of this note is to prove the following theorem.

**Theorem.** If \(Y\) is metric and if \(f(x, y)\) has a measurable modification and \(f(x- y)\) is continuous for almost all \(x\), then \(f(x, y)\) is measurable in both variables together.

This theorem was discovered in an effort to prove that the Nelson canonical version [2] is measurable if it has a measurable modification. The theorem would prove this result except for the restriction that \(Y\) be metric.

---

Received by the editors December 30, 1960 and, in revised form, April 14, 1960.

1 Research supported by the United States Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF 49(638)-265.

Reproduction in whole or part is permitted for any purpose of the United States Government.
We now prove the theorem; without loss of generality we can assume that \( f(x, \cdot) \) is continuous for all \( x \). Let \( f(x, y) \) be a measurable modification. By Lusin's theorem [1], we can find a compact subset \( C \) of \( X \times Y \) on which \( \bar{f}(x, y) \) is continuous and whose measure is greater than \( 1/2 \) the total measure of \( X \times Y \). We can choose \( C \) such that every nonempty relatively open subset has positive measure. Let \( \{ U_n \} \) be a countable basis for the topology of \( Y \). Then in particular \( (X \times U_n) \cap C \) is empty or has positive measure. In either case \( \nu(\{(X \times U_n) \cap C\}) \) is a measurable function of \( x \). Let \( A_n = \{ x ; \nu(\{(X \times U_n) \cap C\}) = 0 \} \). Then \( A_n \) is a measurable subset of \( X \) and \( B_n = \{(A_n \times Y) \cap (X \times U_n) \cap C \} \) has measure zero. Indeed \( (B_n)_x = \emptyset \) if \( x \) is not in \( A_n \) and equals \( \{(X \times U_n) \cap C\}_x \) if \( x \) is in \( A_n \). Hence \( \nu((B_n)_x) = 0 \) for all \( x \) and, since \( B_n \) is a measurable subset of \( X \times Y \), Fubini's theorem implies that it has measure zero. Therefore, \( \bigcup B_n = D \) has measure zero.

Let \( E = C - D \). First we show that \( E_x \) is compact for each \( x \). Let \( N_x = \{ n ; x \in A_n \} \). Then

\[
E_x = (C - D)_x = \left( \bigcap_{n=1}^{\infty} (C - B_n) \right)_x = \bigcap_{n \in N_x} (C_x - (B_n)_x) = \bigcap_{n \in N_x} (C_x - U_n).
\]

This shows that \( E_x \) is compact. If \( E_x \) has positive measure then it is the support of \( \nu \) restricted to it. To see this let \( F_x \) be the support, then \( C_x - F_x \) is relatively open and there exists \( U \subset Y \) such that \( C_x - F_x = U \cap C_x \). Let \( U_n \) be a sequence of sets of \( \{ U_n \} \) whose union is \( U \). Then each \( U_n \) is in \( N_x \) and, therefore, \( U \cap U_n \supset U \). Hence \( E_x \subset F_x \); therefore \( E_x = F_x \).

On \( E \), \( f(x, \cdot) \) is continuous; therefore for each \( x \), \( f(x, \cdot) \) is continuous on \( E_x \) and equals \( \bar{f}(x, \cdot) \) almost everywhere. Now if \( \nu(E_x) \neq 0 \) then Theorem 55.B of [1] implies they are equal everywhere. Hence \( \bar{f}(x, y) = f(x, y) \) for almost all \( (x, y) \), \( [\mu \times \nu] \), in \( E \). An inductive application of this procedure in the complement of \( C \) will yield the theorem.

REFERENCES


Syracuse University