so

$$(y_2y_1' - y_1y_2')(x) = \int_a^x y_2(s)y_1(s)[F(y_2^2(s), s) - F(y_1^2, s)]ds. \tag{7}$$

By hypothesis $(y_2y_1' - y_1y_2')(b) = 0$. The integrand in (7) is positive, however, as long as $y_2(s) > y_1(s)$ and in particular on some interval (a, α)—because $(y_2 - y_1)'(a) > 0$. In fact, α may be taken as b because the same argument as in Lemma 1 shows that the graphs of $y_2(x)$ and $y_1(x)$ can not intersect on (a, b). Thus the right side of (7) does not tend to zero as $x \to b-$. ▌

PROOF OF THE THEOREM. The existence of *at least one* solution of (1)+(C) has been proved by Nehari [1, Theorem IV]. By the preceding lemmas there is *at most one* such solution. ▌

REFERENCE

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ON THE MEASURABILITY OF FUNCTIONS IN TWO VARIABLES

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Let (X, μ) and (Y, ν) be two compact spaces having regular Borel measures defined on them. By a measurable modification $\bar{f}(x, y)$ of a function $f(x, y)$ we mean a function measurable in both variables together and for which $\bar{f}(x\cdot) = f(x\cdot)$ almost everywhere $[\nu]$ for every x. The purpose of this note is to prove the following theorem.

THEOREM. *If Y is metric and if $f(x, y)$ has a measurable modification and $f(x\cdot)$ is continuous for almost all x, then $f(x, y)$ is measurable in both variables together.*

This theorem was discovered in an effort to prove that the Nelson canonical version [2] is measurable if it has a measurable modification. The theorem would prove this result except for the restriction that Y be metric.

Received by the editors December 30, 1960 and, in revised form, April 14, 1960.

[1] Research supported by the United States Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF 49(638)-265.

We now prove the theorem; without loss of generality we can assume that $f(x\cdot)$ is continuous for all x. Let $\tilde{f}(x, y)$ be a measurable modification. By Lusin's theorem [1], we can find a compact subset C of $X\times Y$ on which $\tilde{f}(x, y)$ is continuous and whose measure is greater than 1/2 the total measure of $X\times Y$. We can choose C such that every nonempty relatively open subset has positive measure. Let $\{U_n\}$ be a countable basis for the topology of Y. Then in particular $(X\times U_n)\cap C$ is empty or has positive measure. In either case $\nu(((X\times U_n)\cap C)_x)$ is a measurable function of x. Let $A_n = \{x; \nu(((X\times U_n)\cap C)_x)\}=0$. Then A_n is a measurable subset of X and $B_n=(A_n\times Y)\cap(X\times U_n)\cap C$ has measure zero. Indeed $(B_n)_x=\emptyset$ if x is not in A_n and equals $((X\times U_n)\cap C)_x$ if x is in A_n. Hence $\nu((B_n)_x)=0$ for all x and, since B_n is a measurable subset of $X\times Y$, Fubini's theorem implies that it has measure zero. Therefore, $\bigcup B_n=D$ has measure zero.

Let $E=C-D$. First we show that E_x is compact for each x. Let $N_x=\{n; x\in A_n\}$. Then

$$\begin{aligned} E_x = (C - D)_x &= \left(\bigcap_{n=1}^{\infty} (C - B_n)\right)_x \\ &= \bigcap_{n\in N_x} (C_x - (B_n)_x) \\ &= \bigcap_{n\in N_x} (C_x - U_n). \end{aligned}$$

This shows that E_x is compact. If E_x has positive measure then it is the support of ν restricted to it. To see this let F_x be the support, then C_x-F_x is relatively open and there exists $U\subset Y$ such that $C_x-F_x = U\cap C_x$. Let U_{n_k} be a sequence of sets of $\{U_n\}$ whose union is U. Then each n_k is in N_x and, therefore, $\bigcup_{n\in N_x} U_n\supset U$. Hence $E_x\subset F_x$; therefore $E_x=F_x$.

On $E, f(x\cdot)$ is continuous; therefore for each x, $f(x\cdot)$ is continuous on E_x and equals $\tilde{f}(x\cdot)$ almost everywhere. Now if $\nu(E_x)\neq 0$ then Theorem 55.B of [1] implies they are equal everywhere. Hence $\tilde{f}(x, y)=f(x, y)$ for almost all (x, y), $[\mu\times\nu]$, in E. An inductive application of this procedure in the complement of C will yield the theorem.

References

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