PROJECTIONS ON INVARIANT SUBSPACES[1]

WALTER RUDIN

1. **Introduction.** Let L^1 denote the Banach space of all Lebesgue-integrable functions on the unit circle, and let H^1 be the set of all $f \in L^1$ whose Fourier coefficients $\hat{f}(n)$ are 0 if $n<0$; here

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \qquad (n = 0, \pm 1, \pm 2, \cdots), \tag{1}$$

and the arguments x are understood to be real numbers modulo 2π.

It is clear that H^1 is a closed subspace of L^1. D. J. Newman has shown quite recently [3] that there exist no bounded projections of L^1 onto H^1. (By a *projection* we mean a *linear* operator P such that $P^2=P$.) Certain facts about H^1 suggest a rather general setting for this type of theorem. The abstract problem to which one is thus led turns out to be quite easy, and, when specialized to L^1, yields a result which is much stronger than Newman's (Theorem 2).

Three facts about H^1 are relevant. First, H^1 is *translation-invariant.* That is to say, if the translation operators τ_y are defined by

$$(\tau_y f)(x) = f(x - y) \tag{2}$$

and if $f \in H^1$, then $\tau_y f \in H^1$.

Second, the "natural" projection

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \to \sum_{n=0}^{\infty} c_n e^{inx} \tag{3}$$

is known not to be bounded in the L^1-norm; there exist sequences $\{c_n\}$ such that the series on the left of (3) is the Fourier series of some $f \in L^1$, whereas the series on the right is not [4, p. 253].

Third, if P were a projection of L^1 onto H^1 which commutes with all translation operators τ_y, then P would have to be of the form (3); this is easily verified by considering the action of P on the characters e^{inx}.

Thus there is no bounded projection of L^1 onto H^1 which commutes with all translations, although H^1 is translation-invariant. We shall see that this alone implies that there cannot exist *any* bounded projection of L^1 onto H^1.

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2. For the general setting, let X be a Banach space, let G be a compact topological group, and suppose that G acts as a group of bounded linear operators on X. This means: (a) There is an algebraic isomorphism $g \to T_g$ of G onto a group of bounded linear operators T_g on X (they must be one-to-one and onto); we write gx in place of $T_g x$. (b) The map $(g, x) \to gx$ is a continuous map of the product space $G \times X$ into X.

THEOREM 1. *If X and G are related in this fashion, if Y is a closed subspace of X which is invariant under G (i.e., $gY \subset Y$ for all $g \in G$), and if there exists a bounded projection P of X onto Y, then there exists a bounded projection Q of X onto Y whitch commutes with every $g \in G$.*

Theorems of this type are standard tools in the theory of group representations, and this particular result may very well not be new.

PROOF. For each positive integer k and for each $x \in X$, the set $E_{k,x}$ consisting of all $g \in G$ for which $\|gx\| \leqq k\|x\|$ is closed. If E_k is the set of all $g \in G$ for which $\|g\| \leqq k$, then E_k is the intersection of all sets $E_{k,x}$ and is therefore also closed. Since G is compact, the Baire theorem now implies that $\|g\|$ is bounded on some nonempty open set $V \subset G$, say $\|g\| \leqq B$ on V. If $g \in g_0 V$, then $\|g\| \leqq B\|g_0\|$, and since G is covered by finitely many translates of V, there is a number M such that

$$\|g\| \leqq M \qquad (g \in G). \tag{4}$$

We now define an operator Q on X by

$$Qx = \int_G g^{-1}Pgx\,dg \qquad (x \in X). \tag{5}$$

Here dg denotes the Haar measure of G, normalized so that the measure of G is 1. For every $x \in X$, $g^{-1}Pgx$ is a continuous X-valued function on G. Hence Qx is well defined by (5). It is evident that Q is linear, and (4) shows that $\|Q\| \leqq M^2\|P\| < \infty$.

For any $x \in X$, $Pgx \in Y$ for all $g \in G$, and since Y is invariant under G, $g^{-1}Pgx \in Y$. Since Y is closed, it follows that $Qx \in Y$. Moreover, if $x \in Y$, then $gx \in Y$, $Pgx = gx$, and so $g^{-1}Pgx = x$; hence $Qx = x$. It follows that Q is a bounded projection of X onto Y.

Finally, fix $g_0 \in G$ and put $h = gg_0$, so that $g^{-1} = g_0 h^{-1}$. Since Haar measure is translation-invariant, we have

$$Qg_0x = \int_G g^{-1}Pgg_0x\,dg = \int_G g_0h^{-1}Phx\,dh = g_0Qx.$$

This completes the proof.

3. We now return to L^1. Let S be a closed invariant subspace of L^1, i.e., if $f\in S$ then $\tau_y f\in S$ for all y, where τ_y is defined by (2). Since S is closed, S also contains the functions

$$\hat{f}(n)e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)e^{iny}dy \qquad (n = 0, \pm 1, \pm 2, \cdots). \tag{6}$$

Thus, if $\hat{f}(n)\neq 0$ for some $f\in S$ and for some n, then S contains the character e^{inx}. Since every $f\in L^1$ can be approximated, in the L^1-norm, by finite linear combinations of partial sums of its Fourier series, we conclude that the closed translation-invariant subspaces of L^1 are determined by the characters they contain.

In other words, with every set $N\subset Z$ (where Z is the set of all integers) there is associated a closed translation-invariant subspace S_N of L^1, which consists of all $f\in L^1$ which have $\hat{f}(n)=0$ whenever n is not in N; conversely, every closed translation-invariant subspace of L^1 is one of these spaces S_N.

If Q is a bounded linear operator in L^1 which commutes with every τ_y, then it is easily seen (by studying the action of Q on a single character) that the Fourier coefficients of Qf are of the form $c_n\hat{f}(n)$, where $\{c_n\}$ is a sequence determined by Q. Hence $\{c_n\}$ is a sequence of Fourier-Stieltjes coefficients [**4**, p. 176]. If, in addition, Q is a projection onto S_N, then $c_n=1$ for $n\in N$, $c_n=0$ otherwise, i.e., $\{c_n\}$ is the characteristic function of N. But Helson has shown [**2**] that the only sets $N\subset Z$ whose characteristic functions are Fourier-Stieltjes transforms are the periodic sets (i.e., finite unions of arithmetic progressions, infinite in both directions), and those which differ from periodic sets in finitely many places.

The translation operators τ_y are isometries on L^1, and so

$$\|\tau_y f - \tau_{y_0} f_0\|_1 \leqq \|f - f_0\|_1 + \|\tau_y f_0 - \tau_{y_0} f_0\|_1,$$

which tends to 0 as $f\to f_0$ and $y\to y_0$. Hence the circle acts as a group of bounded linear transformations on L^1, and we can apply Theorem 1. The preceding remarks then immediately yield the following result:

THEOREM 2. *Let N be a subset of Z. There is a bounded projection of L^1 onto S_N if and only if N differs from a periodic set in at most finitely many places.*

In particular, there is no such projection if N is the set of all nonnegative integers; this is Newman's theorem.

4. In place of the integrable functions on the unit circle, we could

equally well have discussed $L^1(G)$, the Banach space of all Haar-integrable functions on a compact abelian group G. We need only to know the idempotent measures on G, i.e., the measures whose Fourier-Stieltjes coefficients are either 0 or 1. Since the theorem of Helson, to which we referred in §3, has been extended to all compact abelian groups by Cohen [1], our discussion can be extended to this case without introducing any new ideas.

Theorem 1 can be applied to any function space on which the operators τ_y act continuously. A particularly simple example is furnished by the space C of all continuous functions on the circle (or on any compact abelian group); the ideas of §3 apply with very minor changes, and we shall merely state the result:

Theorem 2 holds with C in place of L^1, if S_N is interpreted to be the space of all $f \in C$ with $\hat{f}(n) = 0$ outside N.

References

1. P. J. Cohen, *On a conjecture of Littlewood and idempotent measures*, Amer. J. Math. **82** (1960), 191–212.

2. Henry Helson, *Note on harmonic functions*, Proc. Amer. Math. Soc. **4** (1953), 686–691.

3. D. J. Newman, *The nonexistence of projections from L^1 to H^1*, Proc. Amer. Math. Soc. **12** (1961), 98–99.

4. Antoni Zygmund, *Trigonometric series*, 2nd ed., Vol. I, Cambridge Univ. Press, 1959.

University of Wisconsin