ON THE EXISTENCE OF A FUNDAMENTAL DOMAIN FOR RIEMANNIAN TRANSFORMATION GROUPS

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1. **Introduction.** Suppose that a Lie group G acts differentiably on a manifold M. For $x \in M$, $g \in G$, we denote the transform of x by $g \cdot x$ or gx. We say that a subset $D \subset M$ is a *fundamental domain* for the action of G on M if:

1.1. (a) D is an imbedded submanifold of M.

(b) Every element of M can be transformed into $\bar{D}$, the closure of D in M, by the action of G, i.e., $M = G \cdot \bar{D}$. The elements in $G \cdot D$ are said to be *regular*.

(c) For $x \in D$, $Gx \cap \bar{D}$ is empty, i.e., every regular element admits only one representative in $\bar{D}$.

(d) If $\sigma(t)$, $0 \leqq t \leqq 1$, is a differentiable curve on M consisting only of regular elements, there is a differentiable curve $g(t)$, $0 \leqq t \leqq 1$ in G such that $g(t)\sigma(t) \in D$, $0 \leqq t \leqq 1$.

This is just a generalization of the classical concept of fundamental domain, usually formulated for G discrete. In this work we show that fundamental domains with nice properties exist for a large class of nondiscrete transformation groups: We assume that M is a complete, connected Riemannian manifold and that G is a closed subgroup of the group of isometries of M and show that the question can be reduced to asking whether a certain principal fiber bundle admits a sufficiently large cross-section. The technique we use to reduce the problem has other applications to the theory of transformation groups; therefore we devote the later sections to working out some miscellaneous differential-geometric results that will be useful in further work.

2. **Preliminaries.** All manifolds, maps, action of groups, curves, etc., will be differentiability class C^∞ unless mentioned otherwise. We suppose also that G is connected. If $x \in M$, M_x denotes the tangent space to M at x. If $g \in G$, $v \in M_x$, $gv \in M_{gx}$ denotes the tangent vector resulting from applying the transformation g. If $x \in M$: $G^x = \{g \in G: gx = x\}$, *the isotropy subgroup at* x. $Gx = \{gx: g \in G\}$ is the orbit of G at x. It is closed in M. $(Gx)_x$ is the tangent space to the orbit of G at x. P is the union of the principal orbits of G, i.e., the

Received by the editors April 1, 1961.

[1] Lincoln Laboratory, Massachusetts Institute of Technology, operated with support from the U. S. Army, Navy, and Air Force.

orbits of maximal dimension whose isotropy groups have a minimum number of components.

We must use some of the standard results of the theory of transformation groups [1, especially pp. 117–131]. These facts can also be independently derived in a simple way because of our differentiability assumptions:

2.1. *P is a connected, open, dense subset of M.*

2.2. *For $x \in P$, $v \in (Gx)_x^\perp$ (the orthogonal complement to $(Gx)_x$ in M_x), and $g \in G^x$, then we have $gv = v$.*

2.3. *For y sufficiently close to x, G^y is conjugate to a subgroup of G^x. For $x \in M$, let $N^x = \{y \in M: G^x \subset G^y\}$, $N(G^x)$ = the normalizer of G^x in G, $Q^x = N(G^x)/G^x$. The action of $N(G^x)$ passes to the quotient to define an action of Q^x on N^x.*

2.4. *N^x is a totally geodesic submanifold of M [5].*

Suppose now that x is a fixed element of P. Let $N^{x,0}$ be the connected component of N^x containing x. Let $Q^{x,0}$ be the subgroup of Q^x that maps $N^{x,0}$ into itself.

2.5. *The isotropy subgroup of $Q^{x,0}$ at each point of $P \cap N^{x,0}$ is the identity, i.e., $P \cap N^{x,0}$ is a principal fiber bundle, with base space the orbit space of $Q^{x,0}$. $P \cap N^{x,0}$ is open in $N^{x,0}$.*

Proof. Let $g \in N(G^x)$, $y \in N^{x,0}$ and suppose that $gy = y$. Then, $g \in G^y = G^x$, i.e., g, considered as an element of $N(G^x)/G^x$ is the identity. The second part follows from the known fact that the action of a closed group of isometries all of whose orbits are principal defines a fibration over the orbit space.

2.6. *$G \cdot \overline{P \cap N^{x,0}} = M$ and $P \cap N^{x,0}$ is connected.*

Proof. Let $y \in M$. Let σ: $[0, 1] \to M$ be a geodesic of minimal length joining y to Gx. By transforming σ by an element of G, we can suppose that $\sigma(1) = x$. Since σ is perpendicular to Gx, $\sigma(t) \in N^{x,0}$ for $0 \leqq t \leqq 1$, by 2.2 and 2.4. Suppose now that $G^x \neq G^{\sigma(t)}$ for some $t \in [0, 1)$. Let $g \in G^{\sigma(t)} - G^x$. Then, σ and $g\sigma$, joining $\sigma(t)$ to Gx, have the same length and different end-points. σ could not then be a minimizing geodesic from $\sigma(0)$ to Gx. Then, $\sigma(t) \in P \cap N^{x,0}$ for $0 \leqq t < 1$. This argument shows also that $P \cap N^{x,0}$ is connected.

2.7. *If $\sigma(t)$, $0 \leqq t \leqq 1$, is a curve in $G \cdot P \cap N^{x,0}$, then there is a curve $g(t)$, $0 \leqq t \leqq 1$, with $g(t)\sigma(t) \in P \cap N^{x,0}$.*

Proof. It is easy to see that the map $G \times P \cap N^{x,0} \to M$ defining the action of G is of maximal rank. To prove 2.7, we must show that the curve σ in the image can be lifted. The maximal-rank property proves that it can be lifted locally. That it can be lifted globally involves a standard analytic continuation argument, left to the reader.

2.6 and 2.7 together now prove our main result.

THEOREM 2.1. *If D is a fundamental domain for the action of $Q^{x,0}$ on $N^{x,0} \cap P$, then D is a fundamental domain for the action of G on M.*

Notice that a fundamental domain for the action of $Q^{x,0}$ on $N^{x,0} \cap P$ is just a cross-section for the principal fiber bundle defined by the action of $Q^{x,0}$ over an open, dense subset of the base. If $Q^{x,0}$ is discrete, the classical construction of Poincaré defines a fundamental domain for $Q^{x,0}$.

3. The case where $Q^{x,0}$ is discrete.

PROPOSITION 3.1. *If $x \in M$ is such that $Q^x = N(G^x)/G^x$ is discrete, then N^x is perpendicular to every orbit of G that it touches.*

PROOF. For $y \in N^x$, $N^x_y = \{v \in M_y: G^x v = v\}$, which is the direct sum of its intersection with $(Gy)_y$ and $(Gy)_y^{\perp}$. If the intersection with $(Gy)_y$ were nonzero, Q^x could not be discrete. q.e.d.

Let x be a fixed element of P with Q^x discrete. Let $F = N^x - P \cap N^x$. For $y \in F$, let $B^y = \{z \in F: G^y = G^z\}$, a totally geodesic submanifold of N^x.

PROPOSITION 3.2. *If C is a compact subset of M, there are only a finite number of distinct submanifolds among the $\{B^y: y \in F \cap C\}$.*

PROOF. It is known that there are only a finite number of conjugacy classes of subgroups of G that can occur as isotropy subgroups of points of $F \cap C$ [1, p. 110]. If the result to be proved were not true, there would be a point of $y \in F \cap C$ and a sequence of elements (g_n) of G and (y_n) of $F \cap C$ such that:

(a) The (y_n) lie on distinct elements of $\{B^y: y \in F \cap C\}$.

(b) $g_n^{-1} G^y g_n = G^{y_n}$, and

(c) $\lim_n y_n = y$.

We can suppose without any loss of generality that $\lim_n g_n = g_0 \in G$. Since $g_0^{-1} G^y g_0 = G^y$, there is also no loss of generality in assuming that $g_0 =$ identity. Now, (b) implies that $g_n y_n \in B^y \subset N^x$ for all n. But this is a contradiction with the fact that N^x is perpendicular to each orbit of G. q.e.d.

This shows that N^x, for $x \in P$, is a union of a countable, locally finite, system of disjoint, totally geodesic submanifolds, $\{B^y: y \in N^x\}$, i.e., N^x (which is something like a covering space of the space of orbits,

$G\backslash M$) has a geometrically defined "cellular" decomposition. For $y \in N^x$, the map $G \times B^y \to M$ passes to the quotient to define an immersion map $G/G^y \times B^y \to M$; in particular, if M is compact, the singular elements of M break up into the union of a finite number of immersed submanifolds. Any curve lying in these submanifolds can be *smoothly* transformed via G into N^x. That this is a generalization of the classical theorems of differentiable dependence of eigenvalues of matrices varying differentiably should be evident.

4. Some miscellaneous differential-geometric results.

LEMMA 4.1. *Let N be a totally geodesic submanifold of a Riemannian manifold M. Let σ: $[0, 1] \to N$ be a geodesic,*[2] *and let v: $t \to v(t) \in M_{\sigma(t)}$, $0 \leq t \leq 1$, be a Jacobi vector field along σ. Let v_1 (resp. v_2) be the vector field on σ resulting from projecting v parallel (resp. perpendicular) to N. Then, v_1 and v_2 are Jacobi fields.*

PROOF. For the notations and definitions of Riemannian geometry we will use, we refer to [**2**; **3**]. By definition, v satisfies the Jacobi equations.

4.1. $\nabla^2 v(t) + R(\sigma'(t), v(t))(\sigma(t)) = 0$, *where ∇ denotes covariant differentiation along σ, $R(\ ,\ )(\ )$ denotes the curvature tensor.*

Since N is totally geodesic, $\nabla^2 v_1$ (resp. $\nabla^2 v_2$) is tangent (resp. perpendicular) to N and $R(v_1(t), \sigma'(t))(\sigma'(t))$ is tangent to N. To finish the proof that v_1 and v_2 are Jacobi fields, we must show that:

4.2. *If $x \in N$, $v \in N_x^\perp$, $u \in N_x$, then $R(v, u)(u) \in N_x^\perp$.*

Recall the identity:

4.3. $R(u_1, u_2)(u_3) = R(u_1, u_3)(u_2) - R(u_2, u_3)(u_1)$ *for all $x \in M$, all $u_1, u_2, u_3 \in M_x$.*

To prove 4.2, suppose that $u_1 \in N_x$.

$$\begin{aligned}\{R(v, u)(u), u_1\} &= -\{u, R(v, u)(u_1)\} \\ &= -\{u, R(v, u_1)(u)\} + \{u, R(u, u_1)(v)\} \\ &= 0 \text{ since } R(N_x, N_x)(N_x^\perp) \subset N_x^\perp.\end{aligned}$$

($\{\ ,\ \}$ denotes the inner product on M_x defined by the Riemannian metric, $^\perp$ denotes operation of orthogonal complement with respect to this inner product.)

LEMMA 4.2. *Let N be a totally geodesic submanifold of a Riemannian*

[2] Geodesics will be assumed to be parameterized proportionally to arc length.

metric M. Suppose that X is a Killing vector field on M (i.e., an infinitesimal isometry) and that X_1 is the projection of X parallel to N. Then, X_1 is a Killing vector field with respect to the induced metric on N.

PROOF. Recall the condition that X be a Killing field.

4.4. $\{\nabla_v X, v_1\} + \{\nabla_{v_1}, X, v\}$ *for all* $x \in M$, *all* $v_1, v \in M_x$.

But, N totally geodesic implies that the same relation holds for $x \in N$, v, $v_1 \in N_x$ and with X replaced with X_1, since $\nabla_v X_1 \in N_x$, $\nabla_v (X - X_1) \in N_x^{\perp}$. q.e.d.

PROPOSITION 4.3. *Suppose that G is a connected group of isometries of Riemannian manifold M and that N is a complete, immersed, connected totally geodesic submanifold of M that has nonpositive sectional curvature and lies in a compact subset of M. If N is perpendicular to the orbit of G at one point, then it is perpendicular at every point.*

PROOF. Let $x_0 \in N$ be the point at which N is perpendicular to the orbit of G. Let σ: $[0, 1] \to N$ be a geodesic of N with $\sigma(0) = x_0$. Suppose that X is a Killing field on M arising from the action of G. Let X_1 be the Killing field on N that results from projecting X parallel to N. By hypothesis, $X_1(x_0) = 0$. The vector field $t \to X_1(\sigma(t))$ is a Jacobi field of N along σ that is initially zero. If the curvature of N is nonpositive, it is known [**4**, p. 346] that if this field is nonzero it must increase indefinitely in absolute value, which is impossible since X must be bounded on N. Then, $X_1(x) = 0$ for all $x \in N$, which, since X was any infinitesimal G-motion, implies that N is perpendicular to the orbits of G.

This is a generalization of the theorem of Bott and Samelson [**3**, Proposition 2.2] asserting that a geodesic perpendicular to one orbit must be perpendicular to all.

Finally, as an application to the study of the focal points and Morse indices of orbits, we mention the following result, which can be considered as a generalization of some work of Bott and Samelson [**3**] on the quotient of a compact Lie group by a maximal torus.

Let $x \in P$ and suppose that Q^x is discrete. Then, $N_x^x = (Gx)_x^{\perp}$. Let σ: $[0, 1] \to M^x$ be a geodesic transversal to G and with $\sigma(0) = x$. Let $\Lambda(\sigma)$ be the vector-space of all G-transversal Jacobi fields that vanish at $t = 1$. Let $G \cap \Lambda(\sigma)$ be the space of Jacobi fields on σ induced by the action of G vanishing at $t = 1$.

PROPOSITION 5.5. *With the above notations, $\Lambda(\sigma)/G \cap \Lambda(\sigma)$ is isomorphic to the vector space of all Jacobi fields along σ that are tangent to N^x and vanish at $t = 0$ and $t = 1$.*

PROOF. Consider the projection of Jacobi fields along σ parallel to N^x. By Lemma 4.1, $\Lambda(\sigma)$ has an image the space of Jacobi fields tangent to N^x vanishing at $t=0$ and $t=1$. We must show that the kernel of this map is $G\cap\Lambda(\sigma)$. Suppose then that $v\in\Lambda(\sigma)$ is everywhere perpendicular to σ. By definition of G-transversal, there is a geodesic deformation $\delta(s, t)$, $0\leqq s$, $t\leqq 1$, such that: (a) $\delta(0, t)=\sigma(t)$, (b) for each s, $t\to\delta(s, t)$ is a geodesic perpendicular to the orbit G. Let $g(s)$, $0\leqq s\leqq 1$, be a curve in G such that the deformation $\delta_1(s, t)=g(s)\delta(s, t)$ satisfies (a), (b) and: (c) $\delta_1(s, 0)=x$. Then $g(0)=e$. $g'(0)$, the tangent vector to g at $s=0$, can be considered an element of the Lie algebra, $\mathbf{G}$, of G, and also as a Killing vector field X on M. Then one sees that:

$$D_s\delta_1(0, t) = X(\sigma(t)) + v(\sigma(t)) \qquad \text{for } 0 \leqq t \leqq 1,$$

which is everywhere perpendicular to N^x. But, since every G-transversal geodesic starting at x must be in N^x, we must have $\delta_1(s, t)\in N^x$ for $0\leqq s$, $t\leqq 1$. These two conditions are only compatible if:

$$X(\sigma(t)) + v(\sigma(t)) = 0 \qquad 0 \leqq t \leqq 1,$$

which is what we wanted to prove.

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