LINEARLY ORDERABLE SPACES[1]

I. L. LYNN

A *linearly ordered* space $(X, T, <)$ is a set X on which a linear ordering $<$ has been defined such that the interval topology generated by $<$ agrees with T. (A subbasic set for the interval topology induced by $<$ is everything less than or greater than a given point.)

A space (X, T) is *linearly orderable* if there is a linear ordering $<$ on X whose associated interval topology agrees with T.

The subspace $\{-1\} \cup \{n^{-1}: n=1, 2, \cdots\}$ of the real line R is not linearly ordered with respect to the usual ordering of R because $\{-1\}$ is open in the relative topology but not in the interval topology. However, this subspace is homeomorphic to the linearly ordered space $\{n^{-1}: n=1, 2, \cdots\}$, so it is linearly orderable.

The subspace $(0, 1) \cup \{2\}$ of R is, however, not even linearly orderable.

To the best of my knowledge there are no known sufficient conditions for a disconnected space to be linearly orderable. In the connected case the only affirmative results are the well-known characterizations of the arc (cf., e.g. [**2**, p. 168]) and a result of Eilenberg [**1**]. The purpose of this note is to prove the following:

THEOREM. *Every zero-dimensional, separable metrizable space is linearly orderable.*

PROOF. Without loss of generality we may assume our space (X, T) is a subset of the Cantor set C in the unit interval I [**3**, p. 173]. We will construct a homeomorphism f of X into C which maps a countable dense subset of X into a carefully selected subset of those points of C which are end points of intervals of $R \sim C$. We will then show that in $f[X]$:

(a) each point is the T-limit of a monotone increasing sequence of points of $f[X]$, or has an immediate predecessor, or is the minimum of $f[X]$, and

(b) each point is the T-limit of a monotone decreasing sequence of points of $f[X]$, or has an immediate successor, or is the maximum of $f[X]$.

The T and interval topologies coincide precisely when (a) and (b) hold.

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Select a point a_0 in X.

Let I_0 be that closed end third of I containing a_0 and I_2 be the other. If $XI_2 \neq \emptyset$, select a point a_1 in XI_2.

Let I_{00} be that closed end third of I_0 containing a_0 and I_{02} be the other. If $XI_{02} \neq \emptyset$, select a point $a_{1/3}$ in XI_{02}. Similarly let I_{22} be that closed end third of I_2 containing a_1 and I_{20} be the other. If $XI_{20} \neq \emptyset$, select a point $a_{2/3}$ in XI_{20}.

Proceeding inductively, for each integer $n > 1$, define intervals $I_{p_1 \cdots p_n}$ ($p_i = 0$ or 2 for $i = 1, 2, \cdots, n$) and, if $XI_{p_1 \cdots p_n} \neq \emptyset$, a selected point in $XI_{p_1 \cdots p_n}$ subject to the following restrictions:

$I_{p_1 \cdots p_n}$ is a closed end third of $I_{p_1 \cdots p_{n-1}}$.

The selected point in $XI_{p_1 \cdots p_n}$ is the selected point in $XI_{p_1 \cdots p_{n-1}}$ if $p_n = p_{n-1}$.

The selected point in $XI_{p_1 \cdots p_n}$ has index $u \cdot 3^{-n+1} + \sum_{i=1}^{n-1} p_i \cdot 3^{-i}$, where $u = 0$ if $p_n = 0$, and $u = 1$ otherwise.

Note that for each $a \in X$, there is a unique sequence of intervals $\{I_{p_1 \cdots p_n}\}$ such that $\{a\} = \bigcap_{n=1}^{\infty} I_{p_1 \cdots p_n}$. Thus, we may define a mapping $f: X \to C$ by letting $f(a) = \sum_{i=1}^{\infty} p_i \cdot 3^{-i}$.

If a and b are distinct points of X, there is an integer j such that $p_j \neq q_j$. Thus, $\sum_{i=1}^{\infty} p_i \cdot 3^{-i} \neq \sum_{i=1}^{\infty} q_i \cdot 3^{-i}$. So, f is a one-one mapping.

Moreover, f is continuous. For, if $\{b_m\}$ is a sequence of points of X converging to $a \in \bigcap_{n=1}^{\infty} I_{p_1 \cdots p_n}$, then there is an integer N_n such that both b_m and a are in $I_{p_1 \cdots p_n}$ if $m \geq N_n$, whence both $f(b_m)$ and $f(a)$ are in $[\sum_{i=1}^{n} p_i \cdot 3^{-i}, 3^{-n} + \sum_{i=1}^{n} p_i \cdot 3^{-i}]$. A similar argument shows that f^{-1} is also continuous.

We will show next that if a point of $f[X]$ which is an end point of an interval of $R \sim C$ is not a T-limit point from above (below), then it is not a limit point from above (below) in the interval topology. The converse is clear. It is easily seen from the nature of the mapping f that each point of $f[X]$ which is not an end point of an interval of $R \sim C$ is a two-sided T-limit point of $f[X]$. So, this will conclude the proof.

Let $x = \sum_{i=1}^{r} p_i \cdot 3^{-i} \in f[X]$, where $p_r \neq 0$. Thus $p_r = 1$ or 2, and $p_i = 0$ or 2 for $i \neq r$. Clearly x is a T-limit point from at most one side. Thus, by symmetry, it suffices to prove our assertion in case $p_r = 2$ and x is not a T-limit point from above. The details of the following argument may be filled in easily.

Let $y = \inf \{z \in f[X]: x < z\}$. Suppose first that there is a largest integer $n_0 > r$ such that $\alpha = x + 3^{-n_0} \in f[X]$. If there is also a smallest s_0 such that $\beta = x + \sum_{i=n_0+1}^{s_0} 2 \cdot 3^{-i} \in f[X]$, then $y = \beta$, while if there is no such s_0, then $y = \alpha$. If there is no such n_0 and there is a smallest s_0 such that $\gamma = x + \sum_{i=r+1}^{s_0} 2 \cdot 3^{-i} \in f[X]$, then $y = \gamma$, while if there is no such s_0 then $y=3^{-r+1}+\sum_{i=1}^{r-1} p_i\cdot 3^{-i}\in f[X]$. Thus y is the immediate successor of x.

Let $w=\sup\{z\in f[X]: z<x\}$, let $u_j=\sum_{i=1}^{r-j} p_i\cdot 3^{-i}$ for $j=1, 2, \cdots, r-1$, let $u_r=0$, and u_k be that u_j $(j=1, 2, \cdots, r)$ with least index in $f[X]$. If there is a least $n_0\geq r-k+1$ such that $\delta=u_k+3^{-n_0}\in f[X]$, then $w=\delta$, while if there is no such n_0, then $w=u_k$. Thus w is the immediate predecessor of x.

A similar procedure is followed in case $x=0$ or 1.

In a future note the case of arbitrary subspaces of R will be discussed. (See [**4**].)

References

1. S. Eilenberg, *Ordered topological spaces*, Amer. J. Math. **63** (1941), 39–45.
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3. C. Kuratowski, *Topologie*, Vol. I, 4th ed., Warsaw, 1958.
4. I. L. Lynn, *Linearly orderable subspaces of the real line*, Notices Amer. Math. Soc. **8** (1961), 161.

Wayne State University and
Purdue University