CONTINUA WHICH HAVE WIDTH ZERO[1]

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In a recent paper [7], the author showed that with each tree-like continuum there can be associated a non-negative number called the width of M, and it was shown that the plane, E^2, does not contain uncountably many disjoint tree-like continua each having a positive width. This result is used here in establishing some conditions under which a tree-like continuum in E^2 has width zero. There exists a tree-like continuum, such as one which is the sum of a simple triod T and a ray spiralling around T, that has width zero but one of its subcontinua has a positive width. Some of the theorems presented here give conditions under which a tree-like continuum M has width zero hereditarily; that is, every subcontinuum of M has width zero.[2] While such a continuum has a "thinness" property similar to that of a chainable continuum, there do exist in E^2 tree-like continua, as indicated by Anderson [1], which have width zero hereditarily but are not chainable. The question in §4 of [7] and Roberts' result [11] that every chainable continuum has uncountably many disjoint homeomorphic images in E^2 suggest the following questions. If the tree-like continuum M is a subset of E^2 and has width zero hereditarily, does there exist a sequence of disjoint continua in E^2 converging homeomorphically to M? Does a tree-like continuum in E^2 have uncountably many disjoint homeomorphic images in E^2 if it has width zero hereditarily?[3] These questions are not answered, but their converses are direct corollaries to some theorems in [7]. A tree-like continuum M in E^2 has width zero hereditarily either if there exists a sequence of disjoint continua in E^2 converging homeomorphically to M [7, Theorem 5] or if M has uncountably many disjoint homeomorphic images in E^2 [7, Theorem 10].

In this paper, a compact connected metric space is called a *continuum*. Definitions of *trees*, *chains*, *tree-like continua*, and *triods* can be found in [6]. A definition of the *width* of a tree-like continuum is stated in [7], and the following property follows directly from this definition of width. A tree-like continuum M has width zero if, and

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[2] This property was suggested by R. H. Bing during the discussion which followed the presentation of [7] at the Summer Meeting at East Lansing in August, 1960.

[3] It follows from Theorem 5 that this question is equivalent to one raised by Bing [2, p. 656].

only if, for any cofinal sequence $T_1, T_2, \cdots$ of trees defining M, there exists, for each i, a chain C_i in T_i such that the sequence of sets[4] $C_1^*, C_2^*, \cdots$ converges to M.

THEOREM 1. *If M is a tree-like continuum and for every positive number ϵ there is a subcontinuum of M which has width zero and is ϵ-dense[5] in M, then M has width zero.*

PROOF. Let K be a subcontinuum of M which has width zero and is $\epsilon/2$-dense in M. It follows from the above property of continua with width zero that there exists a positive number δ less than $\epsilon/2$ such that every δ-tree which is an essential covering of K must contain a chain C that is $\epsilon/2$-dense[6] in K. Now let G be a δ-tree that is an essential covering of M and let G' be a subtree of G that is an essential covering of K. Hence there is a chain C' in G' that is $\epsilon/2$-dense in K, and it follows that C' is ϵ-dense in M. This implies that M has width zero.

THEOREM 2. *If every proper subcontinuum of the tree-like continuum M has width zero, then M has width zero.*

PROOF. Let ϵ be a positive number, let $p_1, p_2, \cdots, p_n$ be distinct points of M such that every point of M is within a distance $\epsilon/2$ of some p_i, and let $D_1, D_2, \cdots, D_n$ be open sets with disjoint closures such that, for each i, D_i contains p_i and has a diameter less than $\epsilon/2$. Some subcontinuum K of M is irreducible with respect to the property of being a continuum which intersects the closures of all of the sets $D_1, D_2, \cdots, D_n$. It follows from [**4**, Theorem 3] that for some i, K does not intersect D_i. Hence K is a proper subcontinuum of M and is ϵ-dense in M. Since K has width zero, it follows from Theorem 1 that M has width zero.

COROLLARY 2.1. *If every proper subcontinuum of the tree-like continuum M is chainable, then M has width zero.*

COROLLARY 2.2. *If every proper subcontinuum of the tree-like continuum M is an arc, then M has width zero.*

THEOREM 3. *In order that the nondegenerate tree-like continuum M should have width zero, it is necessary and sufficient that M be irreducible between some two points.*

PROOF OF NECESSITY. The continuum M is not a triod [**7**, Theorem

[4] The set which is the sum of the elements of C_i is denoted by C_i^*.

[5] A subset H of M is *ϵ-dense* in M if every point of M is within a distance ϵ of H.

[6] This means that every point of K is within a distance $\epsilon/2$ of some link of C.

3] and is unicoherent [3]. Sorgenfrey [12, Theorem 3.2] has shown that such a continuum is irreducible between some two points.

PROOF OF SUFFICIENCY. Let x and y be two points such that M is irreducible between them, and let $T_1, T_2, \cdots$ be a cofinal sequence of trees defining M. For each i, there is a chain C_i in T_i which covers both x and y. Now some subsequence of $C_1^*, C_2^*, \cdots$ converges to a subcontinuum of M that contains both x and y. But no proper subcontinuum of M contains both x and y, so $C_1^*, C_2^*, \cdots$ converges to M. Hence M has width zero.

COROLLARY 3.1. *Every indecomposable tree-like continuum has width zero.*

COROLLARY 3.2. *Every hereditarily indecomposable tree-like continuum has width zero hereditarily.*

REMARK. Since a tree-like continuum has width zero hereditarily if it is either chainable or hereditarily indecomposable, one might wonder whether the converse is true. However, two continua of a type indicated by Anderson [1] can be joined at a point to obtain a tree-like continuum which is decomposable, has width zero hereditarily, and is not chainable. That every tree-like continuum with these three properties must contain a nondegenerate indecomposable continuum follows from the proof of the next theorem.

THEOREM 4. *If every proper subcontinuum of the tree-like continuum M is decomposable and has width zero, then every proper subcontinuum of M is chainable.*

PROOF. Let K be a proper subcontinuum of M. No triod has width zero [7, Theorem 3], so K contains no triod. That K is hereditarily unicoherent follows from the fact that this is a property of every tree-like continuum [3]. Bing [2] has shown that a continuum is chainable if it is atriodic, hereditarily decomposable, and hereditarily unicoherent. Hence K is chainable.

THEOREM 5. *In order that a tree-like continuum M should have width zero hereditarily, it is necessary and sufficient that M should contain no triod.*

PROOF OF NECESSITY. Every tree-like triod has a positive width [7, Theorem 3]. Hence no subcontinuum of M is a triod.

PROOF OF SUFFICIENCY. Each subcontinuum of M is unicoherent [3] and atriodic. Hence, as in the proof of Theorem 3, it follows from Sorgenfrey's theorem [12, Theorem 3.2] that each subcontinuum of

M is irreducible between some two points. Now by Theorem 3, each subcontinuum of M has width zero.

THEOREM 6. *Every homogeneous*[7] *tree-like continuum in* E^2 *has width zero hereditarily.*

PROOF. Suppose that some homogeneous tree-like continuum M in E^2 does not have width zero hereditarily. It follows from Theorem 2 that some proper subcontinuum K of M does not have width zero. F. B. Jones [9] has shown that every homogeneous tree-like continuum is indecomposable. Hence M has uncountably many disjoint composants,[8] and the homogeneity of M implies that each of these composants contains a homeomorphic image of K. But no homeomorphic image of K has width zero [7, Theorem 2], and this involves a contradiction since E^2 does not contain uncountably many disjoint tree-like continua with positive widths [7, Theorem 10]. Hence M has width zero hereditarily.

THEOREM 7. *If M is a homogeneous continuum in* E^2, *then every proper subcontinuum of M is tree-like and has width zero.*

PROOF. Every homogeneous continuum in E^2 is the boundary of each of its complementary domains [5, Theorem 2]. So each proper subcontinuum of M fails to separate E^2 and hence is tree-like [2]. Now suppose that some proper subcontinuum K of M does not have width zero. The indecomposable case and the decomposable case will be considered separately, and a contradiction will be obtained in each case.

Case 1. If M is indecomposable, then a contradiction can be obtained as in the proof of Theorem 6.

Case 2. If M is decomposable, it follows from a theorem due to F. B. Jones [10] that there is a continuous collection G of homogeneous indecomposable tree-like continua filling M such that the decomposition space of G is a simple closed curve. Hence it follows from Theorem 6 that each element of G has width zero hereditarily, so that K intersects at least two elements of G. Jones [10] has shown that each element of G that intersects K must be a subset of K. Hence there is a continuous subcollection G' of G that fills K so that the decomposition space of G' is an arc. Now it is a further consequence

[7] A continuum M is homogeneous if for each two points x and y of M there is a homeomorphism of M onto itself that carries x into y.

[8] A subset H of M is said to be a composant of M if, for some point p of M, the set H is the sum of all proper subcontinua of M that contain p. Every nondegenerate indecomposable continuum has uncountable many disjoint composants [8].

of Jones' results in [**10**] that K is irreducible from a point in one end element of G' to a point in the other end element of G'. Hence it follows from Theorem 3 that K has width zero.

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