LINEARLY QUASI-ORDERED COMPACT SEMIGROUPS

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By semigroup, we mean a Hausdorff topological space together with a continuous associative multiplication. We are concerned here with the existence of local subsemigroups, which are arcs, in compact connected semigroups whose principal left ideals are linearly ordered by $\subset$. In addition, we seek local cross sections at certain idempotent elements in such semigroups.

In [5], Koch has shown that if S is a compact connected semigroup with identity, then S contains an arc. In [7], Mostert and Shields obtained local one parameter semigroups at the identity. Hunter, in [2], and Hunter and Rothman, in [3], have obtained arcs which are local subsemigroups and certain local cross sections in compact connected abelian topological semigroups. Our approach is similar to that in [2] and [3], and concerns noncommutative semigroups.

We follow the notation and terminology of [1; 2; 3; 11]. In particular, a nonempty subset $L(R, I)$ of a semigroup S is a left (right, two sided) ideal of S if $SL\subset L(RS\subset R,\ SI\cup IS\subset I)$. The left ideal L is principal if for some $x\in L$, $L=\{x\}\cup Sx$. We denote by L_p the set of those x in S for which $\{x\}\cup Sx=\{p\}\cup Sp$. The symbol K is reserved for the minimal ideal of a semigroup (when S is compact, K exists), and E is used to denote the set of idempotent elements of S (when S is compact, E is nonvoid). If $e\in E$, the maximal subgroup of S containing e is designated by H_e.

It is known that the sets L_p form an upper semi-continuous decomposition of S, when S is compact. We let S' be the associated hyperspace of this decomposition and ϕ be the natural mapping. That is $\phi\colon S\to S'$ is given by $\phi(x)=\{L_x\}$. We are interested in the cases when S' is again a semigroup and ϕ is a homomorphism. In particular, Theorem 1 gives necessary and sufficient conditions that S' be a standard thread [1].

DEFINITION. A semigroup S is said to be *left linearly quasi-ordered* if for each x and y in S, either $\{x\}\cup Sx\subset\{y\}\cup Sy$ or $\{y\}\cup Sy\subset\{x\}\cup Sx$.

It is easy to see that the order induced on S, $x\leqq y$ if and only if $\{x\}\cup Sx\subset\{y\}\cup Sy$, is a continuous quasi-order in the sense of Nachbin [8] and Ward [10]. It follows that each compact subset of S has a maximal element.

We assume for the remainder of this note, that S is a compact

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connected semigroup. Since we will concern ourselves only with left linearly quasi-ordered semigroups, we omit the term left. We note that the right linearly quasi-ordered semigroups are dual to the left ones when $xy=yx$.

LEMMA 1. *Let S be a linearly quasi-ordered semigroup. Then* (i) $S^2=S$, (ii) *for each closed left ideal L of S and $y\in S-L$, $L\subset Sy$,* (iii) $S=ES=SE=ESE$, *and* (iv) *for each $x\in S$, $xS\subset Sx$.*

PROOF. (i) Let $y\in S-S^2$, then $y\in S-Sx$ for all $x\in S$. If $x\neq y$, then $\{x\}\cup Sx\subset Sy$ by the linear quasi-order, and hence, $x\in Sy$. It follows that $S^2=S-\{y\}$ is open and closed in S, but S is connected and S^2 is nonvoid. This is a contradiction, hence, $S^2=S$.

(ii) Let L be a closed left ideal of S and $y\in S-L$. For each $x\in L$, $\{x\}\cup Sx\subset L$; hence $\{x\}\cup Sx\subset\{y\}\cup Sy$ by the linear quasi-order and $x\in Sy$, that is $L\subset Sy$.

In order to obtain (iii) and (iv), we show that $eS\subset Se$ for each $e\in E$.

Let $e\in E$ with $eS\not\subset Se$. For $y\in eS-Se$, $Se\subset Sy$ by (ii). Now e and y belong to SeS, hence by a result of Koch [**4**, Corollary 2 to Theorem 3], Se is maximal among $[Sx: x\in SeS]$, hence, $Sy\subset Se$. From this contradiction, it follows that $eS\subset Se$.

(iii) We show first that $S=SE$. If $y\in S-SE$, then for each $e\in E$, $e\in Sy$ and hence, $SeS\subset SyS$. Now $S=SES$ by [**6**, Corollary 1] and $S=SES\subset SyS$. Hence $y=ayb$ for some a and b in S. Thus, there is an $f\in E$ such that $yf=y$ [**9**] and $S=SE$.

In order to see that $S=ES$, we note that for each $f\in E$, $Sf=\bigcup[eS: e\in L_f\cap E]$; since, $S(\bigcup[eS: e\in L_f\cap E])\subset SfS\subset Sf$ and $S(\bigcup[eS: e\in L_f\cap E])$ is a left ideal containing f. Now, $S=SE=\bigcup[Sf: f\in E]=\bigcup(\bigcup[eS: e\in L_f\cap E]: f\in E)=ES$. The equations $ESE=E(SE)=ES=S$ establish the last equality.

(iv) Let $x\in S$ and $y\in xS-Sx$. Then $Sx\subset Sy$ and both x and y belong to SxS by (iii). Again, we apply the result of Koch as above and find that $Sy\subset Sx$. Hence, $xS\subset Sx$.

We observe that for each $x\in S$, $\{x\}\cup Sx=Sx$ and that Sx is a two sided ideal in S. In fact, we have

COROLLARY 1. *Each closed left ideal in S is a principal left ideal and a two sided ideal.*

PROOF. Let L be a closed left ideal of S. Since the quasi-order, $x\leqq y$, is continuous, L has a maximal element x. Now $Sx\subset L$ and for each $y\in L$, $Sy\subset Sx$; hence $Sx=L$. Since $xS\subset Sx$, $SxS\subset Sx$ and Sx is a two sided ideal.

COROLLARY 2. *The minimal ideal K is a minimal left ideal.*

THEOREM 1. *Let S' be the hyperspace of the upper semi-continuous decomposition of S by the sets L_p. Then S' is a semigroup with the multiplication $\{L_x\}\{L_y\}=\{L_{xy}\}$, and a standard thread if and only if S is linearly quasi-ordered.*

PROOF. Let S' be a standard thread and let x and y belong to S with $\phi(x)\leqq\phi(y)$. Then, either $L_x=L_y$ or there is an element b in S with $\phi(x)=\phi(b)\phi(y)$ and $L_x=L_{by}$. Thus $Sx=Sy$ or $Sx=Sby\subset Sy$ and S is linearly quasi-ordered.

On the other hand, let S be linearly quasi-ordered. Since each closed principal left ideal is an ideal, the upper semi-continuous decomposition of S by the sets L_p is the same as the upper semi-continuous decomposition of S by the sets $J_p(x\in J_p$ if and only if $\{x\}\cup Sx\cup xS\cup SxS=\{p\}\cup Sp\cup pS\cup SpS)$. In order to show that S' is a semigroup, it suffices that $(Sx)(Sy)=Sxy$. Now $SxS=Sx$, hence, $Sx\,Sy=Sxy$ by associativity. Since ϕ is a continuous homomorphism, S' is a compact connected semigroup with zero $(=\phi(K))$ and identity $(=\phi(e)$, where $S=Se)$. The order on S' induced by that on S is total and open intervals are open sets. Hence, S' has the order topology and thus is a standard thread.

For the remainder of this paper, we will assume that S' is a standard thread. The following lemma is in part a generalization of a result of Hunter [**2**, Lemma 2], and utilizes his proof.

LEMMA 2. *Let T be a subsemigroup of S'. Suppose that T contains only two idempotent elements and is a standard thread from $\phi(e)$ to $\phi(f)$, where e and f belong to E. Then $\phi^{-1}(T)$ contains a compact connected subsemigroup N such that N modulo $(N\cap L_e)$ is a standard thread from $\{N\cap L_e\}$ to f homeomorphic and isomorphic to T, and $fS-Se=H_f(N-L_e)$.*

PROOF. We restrict our attention to the subsemigroup fS in S. It is easily seen that fS is a compact connected linearly quasi-ordered semigroup with identity f. Let $R=(\phi^{-1}(T)\cap fS)$ modulo $(L_e\cap fS)$. We first show that R is connected. If $R=A\cup B$, where A and B are mutually exclusive compact sets, with $\{L_e\cap fS\}\in B$, then, since A is compact, there is an $x\in A$ such that $\phi(x)$ is the first point of $\phi(A)$ in T in the order from $\phi(e)$ to $\phi(f)$. (We are identifying $(\phi^{-1}(T)\cap fS)-(L_e\cap fS)$ and $R-\{L_e\cap fS\}$.) Let V be an open set such that $A\subset V$ and $V\cap B=\square$. For any open set U containing f, there is a $t\in R$ such that $L_t\cap U\neq\square$, $L_t\neq L_f$. For, let $b\in L_f\cap fS=H_f$, let b^{-1} be the inverse of b in H_f, and let there be sets L_p arbitrarily close to b.

The sets $L_{pb^{-1}}$ are then arbitrarily close to f. By continuity of multiplication, there is an open set W containing f such that $xW\subset V$. Since $xf=x$, $x\in xW$ and there is an L_t such that $t\in L_t\cap W$ and $\phi(t)<\phi(f)$. Since $xL_t\subset L_{xt}$, $L_{xt}\cap V\neq\square$ and $xt\in A$. Since T has only two idempotent elements, $\phi(xt)\neq\phi(x)$. Since $\phi(xt)\leqq\phi(x)$, $\phi(xt)<\phi(x)$ and $\phi(x)$ is not the minimal element of $\phi(A)$. This contradiction shows that R is connected, it is clearly compact and hence a continuum. The semigroup R has a zero $\{L_e\cap fS\}$ and an identity f and no other idempotent elements. It follows from [7] that there is a standard thread P from the zero of R to the identity of R. Letting δ be the natural mapping of $\phi^{-1}(T)\cap fS$ onto R and $J=\delta^{-1}(P-\{0\})$, we see that $N=\bar{J}$ is the desired semigroup.

Let $x\in fS-Se$ and $n\in N\cap L_x$, then $x\in fSn$ and hence $x=ftn$. But $\phi(x)=\phi(n)=\phi(f)\phi(t)\phi(n)$ and thus $\phi(ft)=\phi(f)$. Hence $ft\in H_f$ and $x\in H_f(N-L_e)$. It follows that $fS-Se=H_f(N-L_e)$.

For use in the next lemma, we note that if e and f belong to E with $L_e=L_f$ then $ef=e$ and $fe=f$.

LEMMA 3. *If $f\in E$, then, for each $e\in L_f\cap E$, eS is topologically and algebraically isomorphic to fS under the mapping $ex\rightarrow fex$.*

PROOF. Let $\theta: eS\rightarrow fS$ be defined by $\theta(x)=fx$. If $\theta(x)=\theta(y)$, then $fx=fy$ and $efx=efy$ but then $x=ex=ey=y$ and θ is injective. It is clear that θ is surjective and by its definition continuous, hence θ is a homeomorphism. Now $\theta(xy)=fxy=fxey=fxefy=f(xe)fy=fxfy=\theta(x)\theta(y)$ and θ is a homomorphism.

THEOREM 2. *Let T be a subsemigroup of S'. Suppose that T contains exactly two idempotent elements g' and f' and that $T=[g',f']$ is a standard thread. Then, for each $f\in\phi^{-1}(f')\cap E$, there is a compact connected subsemigroup $N(f)\subset fS$ such that $N(f)$ modulo $(N(f)\cap\phi^{-1}(g'))$ is a standard thread isomorphic to T, and $Sf-Sg=L_f(N(f)-\phi^{-1}(g'))$, where $g\in\phi^{-1}(g')$.*

PROOF. The first part of the theorem is a restatement of Lemma 2. We let f be a fixed idempotent element in $\phi^{-1}(f')$ and $N(f)$ the compact connected semigroup of Lemma 2. For $e\in L_f\cap E$, we choose $N(e)=eN(f)$ by Lemma 3.

Let $y\in Sf-Sg$, then $y\in eS$ for some $e\in L_f\cap E$. Hence $y=pn$, where $p\in H_e$ and $n\in N(e)-\phi^{-1}(g')$ by Lemma 2. Now $pf=p$, so that $y=pfn$ and $y\in L_f(N(f)-\phi^{-1}(g'))$, since $fn\in fN(e)=fefN(f)=N(f)$ and the conclusion follows.

COROLLARY. *Under the same hypotheses as in Theorem 2, for each $x\in\phi^{-1}(T)-\phi^{-1}(g')$, $L_x=L_fx$.*

PROOF. We note that $N(f)$ meets each L set in at most one point and the L sets are disjoint and fill up $\phi^{-1}(T)-\phi^{-1}(g')$. Since $\phi^{-1}(T)-\phi^{-1}(g')=L_f(N(f)-\phi^{-1}(g'))$, the conclusion follows.

For the remainder of this note, we continue to use the hypotheses and notation of Theorem 2. Let us consider the mapping $\delta: L_f\times N(f)\rightarrow Sf$ given by $\delta(p, n)=pn$. If we restrict δ to $(L_f\cap fS)\times N(f)$, then $\delta(p, n)\in fS$. Now, $L_f\cap fS=H_f$ is a compact topological group with identity f. If x and y belong to $N(f)$ and $g\in H_f$; then $gx=gy$ implies $x=y$, since $N(f)$ meets each L set in at most one point. Let V be a neighborhood of f in fS such that, if $x\in V\cap N(f)$ and p and $q\in V\cap H_f$, with $px=qx$, then $p=q$. Then, for any a, b and c in V with $ab=ac$, $b=c$. We see that if such a V exists, then there is an open set W of H_f and an arc $[x, f]\subset N(f)$ such that $\delta|\overline{W}\times[x, f]$ is injective. Further, $\delta|W\times(x, f]$ is a homeomorphism. We have then

THEOREM 3. *There is a local cross section at f in fS if there is a local cancellation semigroup containing f in its interior.*

THEOREM 4. *If H_f is a Lie group, then there is an arc $(x, f]$ in $N(f)$ such that $fS-(\phi^{-1}[0, x])\approx H_f\times(x, f]$.*

PROOF. Let H_f be a Lie group, then, there is an $x\in N(f)$ such that for all $y\in N(f)-xN(f)$ and $p\in H_f$, $py=y$ implies $p=f$. Hence, $\delta|H_f\times(x, f]$ is injective. Using the continuity of the decomposition $\mathfrak{L}$, and the compactness of H_f, it follows that $\delta|H_f\times(x, f]$ is a homeomorphism.

If each He, for $e\in L_f$, is a Lie group, it does not follow that there is an arc $(x, f]$ such that $Sf-Sx\approx L_f\times(x, f]$, for consider

EXAMPLE 1. Let C denote the Cantor set with multiplication $xy=x$. Let I_1 be the usual unit interval and consider $C\times I_1$. If $\mathfrak{D}$ is the upper semicontinuous decomposition of $C\times I_1$ for which $C\times I_1/\mathfrak{D}$ is the Cantor fan then this is again a semigroup and there are no local cross sections at the idempotent elements of the form $(x, 1)$. However since each $L_f\cap fS=\{f\}$ the arcs $N(f)$ are usual unit intervals and the cross sections trivial.

However, we have the

COROLLARY. *If, for each $e\in E$, L_e is a Lie group, then there is a cross section, $L_e\times(x, e]$, at each idempotent element e for which $\phi(e)$ is the right hand end point of a unit or nil thread in S'.*

PROOF. This follows directly from Theorems 2 and 4.

We did not consider the compact connected idempotent subsemigroups of S' due to

EXAMPLE 2. Let R_1 denote the reals modulo 1 and I_3 the idempotent semigroup on the unit interval with multiplication given by $x \cdot y = \min(x, y)$. Let $T = R_1 \times I_3 \times I_3$ and let S be the subsemigroup of T given by $S = (R_1 \times I_3 \times \{0\}) \cup (\{0\} \times \{0\} \times I_3)$. Then S is a linearly quasi-ordered semigroup and S' is isomorphic to I_3. However, the geometric realization of S as a cylinder with a free arc attached shows that there are connected idempotent subsemigroups P in S' such that $\phi^{-1}(P) \neq L_f N(f)$. Here the $N(f)$ exist by [2, Lemma 1].

Further examples can be found in [2] and [3].

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