A NOTE ON THE GENUS OF A KNOT

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For each polygonal knot \( k \) in a 3-sphere \( S^3 \) there exists a polyhedral orientable surface \( M_k \) whose boundary is \( k \) ([1; 2]). The genus of such a knot \( k \), denoted by \( g(k) \) below, was defined by H. Seifert [2] as the minimal number of genera of all such \( M_k \). Now let us define \( \bar{g}(k) \) for each polygonal knot \( k \) as the minimal number of genera of all orientable surfaces \( M_k \) whose boundaries are \( k \), where \( M_k \) may be wildly imbedded. Is \( \bar{g}(k) = g(k) \)? This is a problem proposed by R. H. Fox. The purpose of this short note is to prove that the equality holds.

Proof. Let \( k \) be a polygonal knot in \( S^3 \). Let \( M \) be a polyhedral orientable surface and \( h \) a homeomorphism of \( M \) in \( S^3 \) such that \( h(\partial M) = k \). Then we are only to prove that \( g(M) \), the genus of \( M \), is equal to or greater than \( g(k) \).

Let \( T \) be a tubular neighborhood of \( k \). Bing’s approximation theorem [3] shows that for each \( \varepsilon > 0 \) there exists a semilinear homeomorphism \( h' \) of \( M \) into \( S^3 \) such that for each \( x \in M \) \( d(h(x), h'(x)) < \varepsilon \). If we choose \( \varepsilon \) as to be sufficiently small, we may suppose that \( h'(\partial M) = k' \) is contained in \( T \) and that \( k \) is a companion of \( k' \) with the winding number one in the sense of H. Schubert [4]. By definition \( g(h'(M)) \geq g(k') \). Further H. Schubert [4] shows that if \( k \) is a companion of \( k' \) with the winding number \( \alpha \), then \( g(k') \geq \alpha g(k) + g(k^*) \), where \( k^* \) is a suitably defined polygonal knot. Thus \( g(k') \geq g(k) \) in our case. Therefore \( g(M) = g(h'(M)) \geq g(k') \geq g(k) \), which completes the proof.

References


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