A HAUSDORFF TOPOLOGY FOR THE CLOSED SUBSETS OF A LOCALLY COMPACT NON-HAUSDORFF SPACE

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In the structure theory of C^*-algebras an important role is played by certain topological spaces X which, though locally compact in a certain sense, do not in general satisfy even the weakest separation axiom. This note is concerned with the construction of a compact *Hausdorff* topology for the space $\mathcal{C}(X)$ of all closed subsets of such a space X. This construction occurs naturally in the theory of C^*-algebras; but, in view of its purely topological nature, it seemed wise to publish it apart from the algebraic context.[1]

A comparison of our topology with the topology of closed subsets studied by Michael in [2] will be made later in this note.

For the theory of nets we refer the reader to [1]. A net $\{x_\nu\}$ is *universal* if, for every set A, x_ν is either ν-eventually in A or ν-eventually outside A. Every net has a universal subnet. By the *limit set* of a net $\{x_\nu\}$ of elements of a topological space X we mean the set of those y in X such that $\{x_\nu\}$ converges to y; the net $\{x_\nu\}$ is *primitive* if the limit set of $\{x_\nu\}$ is the same as the limit set of each subnet of $\{x_\nu\}$, i.e., if every cluster point of the net is also a limit of the net. A universal net is obviously primitive. In a locally compact Hausdorff space X the primitive nets are just those which converge either to some point of X or to the point at infinity.

An arbitrary topological space X will be called *locally compact* if, to every point x of X and every neighborhood U of x, there is a compact neighborhood of x contained in U. A compact Hausdorff space is of course locally compact; but a compact non-Hausdorff space need not be locally compact.

Let X be any fixed topological space (no separation axioms being assumed), and let $\mathcal{C}(X)$ be the family of all closed subsets of X (including the void set Λ). For each compact subset C of X, and each finite family $\mathfrak{F}$ of nonvoid open subsets of X, let $U(C; \mathfrak{F})$ be the subset of $\mathcal{C}(X)$ consisting of all Y such that (i) $Y \cap C = \Lambda$, and (ii) $Y \cap A \neq \Lambda$ for each A in $\mathfrak{F}$. A subset $\mathcal{W}$ of $\mathcal{C}(X)$ is *open* if it is a union of certain of the $U(C; \mathfrak{F})$. It is easily verified that this notion of openness defines a topology for $\mathcal{C}(X)$, which we will call the *H-topology.*

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LEMMA 1. *$\mathcal{C}(X)$ is compact in the H-topology.*

PROOF. Let $\{Y^i\}$ be a universal net of elements of $\mathcal{C}(X)$, and define Z to be the set of those x in X such that, for each neighborhood U of x, $Y^i \cap U \neq \Lambda$ for all large enough i. Obviously $Z \in \mathcal{C}(X)$. It will be sufficient to show that $Y^i \to Z$ in the H-topology.

Let $U(C; \mathfrak{F})$ be a typical neighborhood of Z (C and $\mathfrak{F}$ being as before). For each A in $\mathfrak{F}$ there is an element x of $Z \cap A$; and the definition of Z gives

(1) $$Y^i \cap A \neq \Lambda \text{ for all large enough } i.$$

Now suppose it were false that

(2) $$Y^i \cap C = \Lambda \text{ for all large enough } i.$$

Then by the universality of $\{Y^i\}$, $Y^i \cap C \neq \Lambda$ for all large enough i. Choosing an element x_i of $Y_i \cap C$ for each large enough i, we have by the compactness of C (passing to a subnet if necessary) $x_i \to z$ for some z in C. So, for each neighborhood V of z, $Y^i \cap V \neq \Lambda$ for all large enough i; whence $z \in Z$, or $C \cap Z \neq \Lambda$. This is impossible since $Z \in U(C; \mathfrak{F})$; so (2) is proved. By (1) and (2) and the arbitrariness of $U(C; \mathfrak{F})$, we have $Y^i \to Z$.

THEOREM 1. *If X is locally compact, $\mathcal{C}(X)$ with the H-topology is a compact Hausdorff space.*

PROOF. Let Y_1 and Y_2 be distinct elements of $\mathcal{C}(X)$, and suppose $x \in Y_1 - Y_2$. By local compactness there is a compact neighborhood V of x for which $V \cap Y_2 = \Lambda$; thus $Y_2 \in U(V; \Lambda)$. Clearly $Y_1 \in U(\Lambda; \{V'\})$, where V' is the interior of V. Since $U(V; \Lambda)$ and $U(\Lambda; \{V'\})$ are disjoint, Y_1 and Y_2 have disjoint neighborhoods. So $\mathcal{C}(X)$ is Hausdorff. It is compact by Lemma 1.

It may be well at this point to contrast our H-topology with the "finite topology" of Michael [2, p. 153]. The latter is defined by a basis consisting of sets $U(C; \mathfrak{F})$ similar to ours except that the C are required to be closed instead of compact. This makes a considerable difference in the properties of the two topologies. The properties of the "finite topology" parallel those of X more closely than do those of the H-topology. For example, the "finite topology" is Hausdorff if and only if X is regular [2, Theorem 4.9.3]. On the other hand, the H-topology is Hausdorff whenever X is locally compact, no matter how badly unseparated X may be; while if X is not locally compact, it may even be metrizable (say an infinite-dimensional Banach space) without the H-topology being Hausdorff. Again, if X is locally compact and T_1 but not Hausdorff, the map $x \to \{x\}$ will not be a homeomorphism with respect to the H-topology, so that the latter is not "admissible" in the sense of [2, p. 153]. Needless to say, if X is compact and Hausdorff, the "finite" and H-topologies coincide.

We shall henceforth always assume that X is locally compact, and that $\mathcal{C}(X)$ is equipped with the H-topology. Here are a few remarks on the H-topology, whose proof is left to the reader:

(I) The operation of union (carrying A, B into $A\cup B$) is continuous on $\mathcal{C}(X)\times\mathcal{C}(X)$ into $\mathcal{C}(X)$. Not so with intersection, however.

(II) If $Y\in\mathcal{C}(X)$, the topology of $\mathcal{C}(Y)$ relativized from $\mathcal{C}(X)$ is the H-topology of $\mathcal{C}(Y)$.

(III) Let α be an infinite cardinal. If X has a basis for its open sets of cardinality no greater than α, then $\mathcal{C}(X)$ has a basis for its open sets of cardinality no greater than α.

(IV) If X is a locally compact topological *group*, the family $\mathcal{S}$ of all closed subgroups of X is a closed subfamily of $\mathcal{C}(X)$. Thus the H-topology relativized to $\mathcal{S}$ is compact and Hausdorff.

We shall now consider an important subset of $\mathcal{C}(X)$. For each x in X, let $\langle x\rangle$ denote the closure in X of the one-element set $\{x\}$. Let $\mathcal{H}(X)$ be the closure in $\mathcal{C}(X)$ of the set of all $\langle x\rangle$, where x ranges over X. By Theorem 1, $\mathcal{H}(X)$ is a compact Hausdorff space, and the image of X under the map $x\rightarrow\langle x\rangle$ is dense in $\mathcal{H}(X)$.

We refer to $\mathcal{H}(X)$ as the *Hausdorff compactification* of X. For obtaining its topology the following lemma is useful:

LEMMA 2. *Let $\{x_\nu\}$ be a net of elements of X, and Y a closed subset of X. The following two conditions are equivalent*:

(i) *$\{x_\nu\}$ is primitive, and Y is its limit set*;

(ii) $\lim_\nu \langle x_\nu\rangle = Y$ *in* $\mathcal{C}(X)$.

PROOF. Assume (i); and let $U(C;\mathfrak{F})$ be a neighborhood of Y. If $A\in\mathfrak{F}$ and $x\in A\cap Y$, then $x_\nu\rightarrow x$, so

$$\langle x_\nu\rangle \cap A \neq \Lambda \quad \nu\text{-eventually.} \tag{3}$$

Suppose it is false that

$$\langle x_\nu\rangle \cap C = \Lambda \quad \nu\text{-eventually.} \tag{4}$$

Then there is a subnet $\{y_\mu\}$ of $\{x_\nu\}$ such that $\langle y_\mu\rangle\cap C\neq\Lambda$; let $z_\mu\in\langle y_\mu\rangle\cap C$. Passing again to a subnet, we may assume $z_\mu\rightarrow z$, $z\in C$. Now each open neighborhood of z contains z_μ, hence intersects $\langle y_\mu\rangle$, hence contains y_μ, for all large enough μ. Thus $y_\mu\rightarrow z$; so that by the primitivity of $\{x_\nu\}$ we have $z\in Y$, $Y\cap C\neq\Lambda$. This contradicts the fact that $Y\in U(C;\mathfrak{F})$; so (4) is proved. Now (3) and (4) and the arbitrariness of $U(C;\mathfrak{F})$ establish (ii).

Now assume (ii). Let $y_\mu\rightarrow y$, where $\{y_\mu\}$ is a subnet of $\{x_\nu\}$; and suppose $y \notin Y$. Then there is a compact neighborhood W of y such that $Y \in U(W; \Lambda)$. By (ii) $y_\mu \notin W$ for all large enough μ, which is impossible since $y_\mu \to y$. Therefore Y contains every cluster point of $\{x_\nu\}$. We shall complete the proof of (i) by showing that Y is contained in the limit set of $\{x_\nu\}$. Let y be in Y, and A be any open neighborhood of y. Then $Y \in U(\Lambda; \{A\})$; so that by (ii) $x_\nu \in A$ for all large enough ν. Thus $x_\nu \to y$.

COROLLARY. *The elements of* $\mathcal{H}(X)$ *are precisely the limit sets of primitive nets of elements of* X.

Here are a few easily verified examples of the Hausdorff compactification:

(I) If X is compact and Hausdorff, $\mathcal{H}(X)$ and X are the same (if we identify x and $\{x\}$). If X is locally compact and Hausdorff but not compact, $\mathcal{H}(X)$ is the one-point compactification of X (the void set in $\mathcal{C}(X)$ being the point at infinity).

(II) Let X be the closed interval $[0, 1]$ together with an extra "zero" $0'$. A subset A of X is to be open if (i) $A \cap [0, 1]$ is open in the usual sense, and (ii) if $0' \in A$, then A contains the open interval $(0, \epsilon)$ for some $\epsilon > 0$. This defines a locally compact non-Hausdorff topology for X. $\mathcal{H}(X)$ consists of the one-element sets together with the two-element set $\{0, 0'\}$; it is homeomorphic with the space S consisting of the ordinary Euclidean closed interval $[0, 1]$ together with two isolated points p and q. The homeomorphism is implemented by the map of $\mathcal{H}(X)$ onto S which sends $\{r\}$ into r for $0 < r \leqq 1$, $\{0, 0'\}$ into 0, $\{0\}$ into p, and $\{0'\}$ into q.

(III) Let X be the square $\{(x, y) \mid 0 \leqq x \leqq 1, 0 \leqq y \leqq 1\}$. For each $\epsilon > 0$, let $B_\epsilon = X \cap \{(x, y) \mid 0 < y < \epsilon\}$. Let a subset A of X be called open if (i) A is open in the usual Euclidean topology of X, and (ii) if $(0, 0) \in A$, then $B_\epsilon \subset A$ for some $\epsilon > 0$. This makes X a locally compact non-Hausdorff space. The elements of $\mathcal{H}(X)$ are the one-element sets, and also the two-element sets of the form $\{(0, 0), (x, 0)\}$ for $0 < x \leqq 1$. $\mathcal{H}(X)$ is homeomorphic with the subspace S of the ordinary Euclidean plane consisting of the square X together with the closed line segment from $(-1, 0)$ to $(0, 0)$. The homeomorphism is implemented by the map of $\mathcal{H}(X)$ onto S which sends $\{(x, y)\}$ into (x, y) for $y \neq 0$, $\{(0, 0), (x, 0)\}$ into $(x, 0)$ for $0 < x \leqq 1$, and $\{(x, 0)\}$ into $(-x, 0)$ for $0 \leqq x \leqq 1$.

It may be of some interest to give an intrinsic characterization of the Hausdorff compactification. As always, X is a fixed locally compact topological space. Let K be a fixed regular topological space.

DEFINITION. A function f on X to K will be called *quasi-continuous* if, for each primitive net $\{x_\nu\}$ of elements of X, $\lim_\nu f(x_\nu)$ exists in K.

If X is locally compact and Hausdorff, it is easily seen that a function f on X to K is quasi-continuous if and only if (i) f is continuous in the usual sense on X, and (ii) $f(x)$ approaches a limit in K as x approaches the point at infinity.

Suppose now that g is a continuous function on $\mathcal{H}(X)$ to K. For each x in X, define

$$f(x) = g(\langle x \rangle). \tag{5}$$

By Lemma 2 f is quasi-continuous on X. Conversely, suppose that f is quasi-continuous on X to K. If $Y \in \mathcal{H}(X)$, by the corollary to Lemma 2 there is a primitive net $\{x_\nu\}$ of elements of X with Y as its limit set. By quasi-continuity, $\lim_\nu f(x_\nu)$ exists in K; and it is easy to see that $\lim_\nu f(x_\nu)$ depends only on Y. Let us define $g(Y) = \lim_\nu f(x_\nu)$. Then g is a function on $\mathcal{H}(X)$, and (5) holds. In fact, using the regularity of K we see without difficulty that g is continuous on $\mathcal{H}(X)$. We therefore have:

THEOREM 2. *Let X be a locally compact topological space, with Hausdorff compactification $\mathcal{H}(X)$; and let K be any regular topological space. Then* (5) *sets up a one-to-one correspondence $f \leftrightarrow g$ between the set of all quasi-continuous functions f on X to K and the set of all continuous functions g on $\mathcal{H}(X)$ to K.*

Let us specialize K to be (for example) the complex number system. Then the property stated in Theorem 2 uniquely describes $\mathcal{H}(X)$. More precisely:

THEOREM 3. *Let X be as in Theorem 2, $\mathcal{Z}$ a compact Hausdorff space, and Φ a mapping of X onto a dense subset of $\mathcal{Z}$ such that the quasi-continuous complex functions on X are precisely the $g \circ \Phi$, where g runs over the continuous complex functions on $\mathcal{Z}$. Then there exists a homeomorphism F of $\mathcal{Z}$ onto $\mathcal{H}(X)$ such that*

$$F(\Phi(x)) = \langle x \rangle \qquad (x \in X).$$

PROOF. By Theorem 2, the space Q of quasi-continuous complex functions on X forms a commutative Banach algebra whose maximal ideal space is homeomorphic to $\mathcal{H}(X)$. But, according to the hypothesis of Theorem 3, it is also homeomorphic to $\mathcal{Z}$. Thus $\mathcal{Z} \cong \mathcal{H}(X)$; and $\Phi(x)$ and $\langle x \rangle$ give corresponding maximal ideals.

BIBLIOGRAPHY

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