CONVERGENCE OF APPROXIMATING POLYNOMIALS

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I. The problem we wish to consider is the following. For each positive integer n, let E_n be a finite subset of $[-1, 1]$ containing at least n points. For a real-valued continuous function f defined on $[-1, 1]$ let $p_n(f, E_n)$ be the unique polynomial of degree at most $n-1$ of best approximation in the Chebycheff sense to f on E_n. Is it possible to choose a fixed sequence $\{E_n\}$ so that for each f, continuous on $[-1, 1]$, $p_n(f, E_n)$ converges to f uniformly on $[-1, 1]$?

A classical result of Faber [4] states that if, for each n, E_n contains exactly n points, this choice is never possible. In this case, of course, $p_n(f, E_n)$ is just the polynomial which interpolates to f at the points of E_n.

In this paper we shall prove that the result of Faber still holds if each E_n contains no more than $n+1$ points. On the other hand, letting $\|f\| = \sup_{-1\leq t\leq 1} |f(t)|$, we obtain $\|f-p_n(f, E_n)\| \to 0$ for each f continuous on $[-1, 1]$, if and only if there exists a constant K independent of n, such that for each polynomial $p_n(x)$ of degree at most $n-1$, if $|p_n(x)| \leqq 1$ for each $x \in E_n$, then $\|p_n\| \leqq K$.

The existence of such sets E_n was first proved by Bernstein [1, pp. 55–57]. In fact $E_n = \{\cos(k\pi/m)\}$, $k=0, 1, \cdots, m$, where $m/n > \pi/2\cdot 2^{1/2}$ is a simple example. It is further shown in [1] that for each fixed $\lambda > 1$ if k_n satisfies $k_n/n > \lambda$ then we may choose a sequence $\{E_n\}$ with the desired properties and such that the cardinality of $E_n = k_n$. Namely, assuming $k_n \leqq 2n$, let E_n consist of the points $\cos((2k-1)/2n)\pi$, $k=1, \cdots, n$, together with the points $\cos(lt\pi/n)$ where l is an integer satisfying $k_n - n - 1 = [n/l]$ and $t = 0, 1, \cdots, [n/l]$.

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II. Let $C = C[-1, 1]$, the Banach space of real-valued continuous functions on $[-1, 1]$ provided with the norm $\|f\| = \sup_{-1\leq t\leq 1} |f(t)|$. Let H_n be the n dimensional sub-space of polynomials of degree $n-1$. Denote by P_n the mapping $f \to p_n(f, E_n)$. P_n is a continuous mapping of C onto H_n satisfying $P_m P_n = P_n$ for $m \geqq n$. In general, P_n is not linear, but if E_n contains either n or $n+1$ points, then P_n is linear which is the crucial fact needed in the following:

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THEOREM 1. *For $n=1, 2, \cdots$ fix a sequence of finite subsets E_n of $[-1, 1]$. If each E_n contains either n or $n+1$ points, then there exists an $f \in C$ to which $p_n(f, E_n)$ fails to converge uniformly on $[-1, 1]$.*

PROOF. Assume for the moment that P_n is linear for each n. Let $f \in H_n$. Then if $m>n$, $P_m(f)=f$ since P_n maps C onto H_n and $P_mP_n = P_n$. By the Weierstrass approximation theorem the polynomials are dense in C, hence we may infer from the principle of uniform boundedness [3, Theorem II.3.6] that $P_n(f) \to f$ for each $f \in C$ iff $\sup_n \|P_n\| < \infty$, where $\|P_n\| = \sup_{\|f\| \leq 1} \|P_n(f)\|$. But by [5, Hilfssatz 3, p. 495] if P_n is any bounded projection of C onto H_n, $\|P_n\| \geqq \ln(n-1)/8\pi^{1/2}$.

Now P_n is clearly linear if E_n contains exactly n points. If E_n contains $n+1$ points x_i, $-1 \leqq x_1 < x_2 < \cdots < x_{n+1} \leqq 1$, let $q_{n+1}(x) = \sum_{k=0}^{n} a_k x^k$ and $r_{n+1}(x) = \sum_{k=0}^{n} b_k x^k$ be the unique polynomials determined by the conditions $q_{n+1}(x_i) = (-1)^i$, $r_{n+1}(x_i) = f(x_i)$, $i=1, 2, \cdots, n+1$. It is easily seen by considering the determinants involved that $a_n \neq 0$. Therefore, let $p_n(x) = r_{n+1}(x) - (b_n/a_n)q_{n+1}(x)$. The mapping $f \to p_n$ is clearly linear, since $f \to r_{n+1}$ and $f \to b_n$ are both linear. But $f(x_i) - p_n(x_i) = (b_n/a_n)(-1)^i$, $i = 1, 2, \cdots, n+1$. Therefore, $p_n = p_n(f, E_n)$ by the classical result of de la Vallée Poussin [2] which completes the proof.

We note two facts. First, it may be easily verified that if E_n contains more than $n+1$ points, P_n is never linear. Secondly, if $q_n(f, E_n)$ denotes the polynomial of best approximation to f on E_n in the sense of least squares, then the same argument as above shows that if $\{E_n\}$ is any sequence of finite subsets of $[-1, 1]$ containing at least n points, then for some $f \in C$, $q_n(f, E_n)$ fails to converge uniformly to f. This follows since the mapping $f \to q_n(f, E_n)$ is always linear and idempotent.

III. We now prove the convergence criterion.

THEOREM 2. *For each $n>0$ let E_n be a finite subset of $[-1, 1]$. Then $\|f - p_n(f, E_n)\| \to 0$ for each $f \in C$ iff there exists a constant K such that if $p \in H_n$, $|p(x)| \leqq 1$, $x \in E_n$, then $\|p\| < K$.*

PROOF. This is a theorem of uniform boundedness type, and although the operators P_n are nonlinear the proof resembles that for the linear case.

With no loss in generality, we assume each E_n contains at least $n+1$ points. For fixed E_n and $p \in H_n$ let $\delta(p) = \sup_{x \in E_n} |f(x) - p(x)|$. Then by a well-known result of de la Vallée Poussin [2] $p_n = p_n(f, E_n)$ is characterized uniquely by the condition that there exist $n+1$ points x_i in E_n, $x_i \leqq x_{i+1}$, for which either

$$f(x_i) - p_n(x_i) = (-1)^i\delta(p_n), \qquad i = 1, 2, \cdots, n+1,$$

or

$$f(x_i) - p_n(x_i) = (-1)^{i+1}\delta(p_n), \qquad i = 1, 2, \cdots, n+1.$$

From this it follows easily that the operator P_n is homogeneous, and if q is a polynomial of degree $<n$, then for each $f \in C$, $P_n(f+q) = P_n(f) + P_n(q) = P_n(f) + q$. Moreover E_n satisfies the condition of the theorem iff $\sup_n \|P_n\| < \infty$. For, by the above remarks, if $p \in H_n$ and $|p(x)| \leqq 1$, $x \in E_n$, then $p = P_n(f, E_n)$ for some f, $\|f\| \leqq 2$. Conversely, if $\|f\| \leqq 1$, then $|p_n(x)| \leqq 2$ for $x \in E_n$ for otherwise $p(x) \equiv 0$ would provide a better approximation on E_n. Therefore, suppose $\sup_n \|P_n\| \equiv K < \infty$. For each $\epsilon > 0$ choose a polynomial q such that $\|f - q\| < \epsilon$. If n_ϵ is the degree of q and $n > n_\epsilon$, then

$$\begin{aligned} \|f - P_n(f)\| &\leqq \|f - q\| + \|q - P_n(f)\| \\ &= \|f - q\| + \|P_n(q - f)\| \leqq \epsilon(1 + K). \end{aligned}$$

Therefore, $\|f - P_n(f)\| \to 0$ and the condition is sufficient.

Conversely, suppose $\|f - P_n(f)\| \to 0$ for each $f \in C$. Since each P_n is continuous, $S_{n,k} \equiv \{f \in C : \|P_n(f)\| \leqq k\}$ is a closed subset of C. Therefore, by the Baire category theorem, for some $k > 0$, $S_k \equiv \bigcap_{n=1}^{\infty} S_{n,k}$ contains an open set. Consequently, there exists a polynomial $q(x)$ and a positive number δ such that if $\|f\| < \delta$, then $f + q \in S_k$. Hence, for $n >$ degree of q and $\|f\| < \delta$,

$$\|P_n(f)\| \leqq \|P_n(q)\| + \|P_n(f - q)\| \leqq \|q\| + k.$$

Using this and the continuity of each P_n it follows that

$$\sup_n \|P_n(f)\| < \infty, \qquad \|f\| < \delta,$$

and the theorem is proved.

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