A REAL INVERSION FORMULA FOR A CLASS OF CONVOLUTION TRANSFORMS

CHARLES STANDISH

In this paper an inversion formula involving only values of $f(x)$ along the real axis is obtained for the class of convolution transforms described below. A real inversion formula involving $f(x)$ and its derivatives was obtained by H. Pollard [1] and a complex inversion formula by Hirschman and Widder [2].

We shall prove the following

THEOREM. *Let*

$$f(x) = \int G(x - t)\phi(t)dt, \tag{1}$$

where

$$\int G(t) \exp(-ist)dt = \left[\prod_{k=1}^{\infty} (1 + s^2/a_k^2) \right]^{-1} = F(is)$$

and

$$\sum_{k=1}^{\infty} (a_k^2)^{-1} < \infty, \qquad 0 < a_k \leqq a_{k+1}, \qquad k = 1, 2, \cdots.$$

If the integral (1) *converges for a single real x, it converges for all real x and is inverted by*

$$\lim_{t\to 0+} \lim_{n\to\infty} \int f(\xi)d\xi \int \prod_{k=1}^{n} (1 + u^2/a_k^2) \cdot\exp[-tu^2 + iu(x - \xi)]du = \phi(x) \tag{2}$$

for almost every x.

It is assumed that $\phi(t)$ is Lebesgue integrable on every finite interval and the integral in (1) is interpreted as

$$\lim_{R\to\infty,\ S\to\infty} \int_{-S}^{R} .$$

When limits are omitted from an integral appearing in the text, the

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range of integration is understood to be $(-\infty, \infty)$.

It is convenient to decompose the proof of the theorem into several lemmas.

LEMMA 1. *If the transform* (1) *converges for a single real* x, *it converges for all real* x.

This was established in [1]. We note that $G(t)$ is a class I kernel in the sense of Widder and Hirschman [2, p. 120].

LEMMA 2.

$$(2\pi)^{-1}\int F(is)\exp its\,ds$$

exists and equals $G(t)$.

By [2, p. 52], $F(is)=O(|s|^{-p})$ for every p, hence is in $L(-\infty, \infty)$ from which the result follows immediately.

LEMMA 3.

$$\int f(\xi)d\xi\int\prod_{k=1}^{n}(1+u^2/a_k^2)\exp(-tu^2-iu(x-\xi))du$$

exists.

$f(\xi)=o(\exp(a_1|\xi|))$ by [2, Theorem 2.1, p. 147]. The inner integral is easily shown by a direct evaluation to be $O(\exp(-A(x-\xi)^2))$, $0<A<(4\epsilon)^{-1}$, which establishes the result.

LEMMA 4.

$$f_A(\xi)=\int_{-A}^{A}G(\xi-s)\phi(s)ds=O(\exp(a_1|\xi|))$$

independent of A.

From [2, p. 123] $G(x-t)/G(-t)$ is nondecreasing or nonincreasing as a function of t according as x is greater or less than zero. The asymptotic estimates furnished by [2, Theorem 2.1, p. 108] show furthermore that

$$\lim_{A\to\infty}G(x-A)/G(-A)=\exp a_1x,$$

$$\lim_{A\to-\infty}G(x-A)/G(-A)=\exp(-a_1x).$$

In the case $x>0$, the mean value theorem enables us to write:

$$f_A(x) = \int_{-A}^{A} [G(x-t)/G(-t)]G(-t)\phi(t)dt$$
$$= G(x-A)/G(-A)\int_{\xi}^{A} G(-t)\phi(t)dt,$$

hence

$$|f_A(x)| \leqq B \exp a_1 x,$$

where

$$B = \sup_{\text{all } R,S} \int_R^S G(-t)\phi(t)dt.$$

A similar argument applies in the case $x<0$.

LEMMA 5.

$$\int f(\xi)d\xi \int \prod_{k=1}^{n} (1 + u^2/a_k^2) \exp(iu(x+\xi) - tu^2)du$$
$$= \int \phi(s)ds \int \prod_{k=n+1}^{\infty} \{(1+u^2/a_k^2)\}^{-1} \exp(iu(x-s) - tu^2)du. \tag{3}$$

The technique employed in the proof of this lemma is the one employed by Blackman [4] in his treatment of convolutions with rational kernels. We have by the definition of $f(x)$, (3) is equal to

$$\int d\xi \left[\lim_{A\to\infty} \int_{-A}^{A} G(\xi - s)\phi(s)ds \right] G_n(x-\xi), \tag{4}$$

where $G_n(x-\xi)$ is the value of the inner integral in (3). The estimate of Lemma 4 enables us to take the limit outside the outer integral whence (4) becomes

$$\lim_{A\to\infty} \int G_n(x-\xi)d\xi \int_{-A}^{A} G(\xi - s)\phi(s)ds. \tag{5}$$

By Fubini's theorem, we can interchange the order of integration which replaces (5) by

$$\lim_{A\to\infty} \int_{-A}^{A} \phi(s)ds \int G_n(x-\xi)G(\xi-s)d\xi.$$

A straightforward calculation shows the inner integral above can be expressed as

$$\int \left[\prod_{k=n+1}^{\infty} (1 + u^2/a_k^2) \right]^{-1} \exp\,(iu(x - s) - tu^2)du,$$

completing the proof of the lemma.

LEMMA 6.

$$\lim_{n\to\infty} \int \phi(s)[H_n(x - s) - (4\pi t)^{-1/2} \exp(-(x - s)^2/4t)]ds = 0,$$

where

$$H_n(x) = \int \left[\prod_{k=n+1}^{\infty} (1 + u^2/a_k^2) \right]^{-1} \exp\,(iux - tu^2)du.$$

$\int\phi(s) \exp(-(x-s)^2/4t)ds$ exists for every real x. Given $\epsilon>0$, we can for any fixed x choose R so large that

$$\sup_{R<S<\infty} (4\pi t)^{-1/2}\left|\int_R^S \phi(s) \exp\,(-(x - s)^2/4t)ds\right| < \epsilon,$$

$$\sup_{R<S<\infty} (4\pi t)^{-1/2}\left|\int_{-S}^{-R} \phi(s) \exp\,(-(x - s)^2/4t)ds\right| < \epsilon.$$

It is easily established that:

$$\begin{aligned} d/dx(H_n(x)/\exp(-x^2/4t)) &\geqq 0, \qquad x \geqq 0, \\ &\leqq 0, \qquad x \leqq 0. \end{aligned}$$

By the mean value theorem, if $R>x$,

$$\begin{aligned} &\left|\int_R^\infty \phi(s)H_n(x - s)ds\right| \\ &\leqq [H_n(x - R)/\exp(-(x - R)^2/4t)] \times \left|\int_R^\xi \phi(s) \exp(-(x - s)^2/4t)ds\right|, \end{aligned} \tag{6}$$

$$\begin{aligned} \left|\int_{-\infty}^{-R} \phi(s)H_n(x - s)ds\right| &\leqq [H_n(x + R)/\exp(-(x + R)^2/4t)] \\ &\quad\times \left|\int_\xi^{-R} \phi(s) \exp(-(x - s)^2/4t)ds\right|. \end{aligned} \tag{7}$$

$H_n(x)$ tends boundedly to $(4\pi t)^{-1/2} \exp(-x^2/4t)$ enabling us to conclude that as $n\to\infty$

$$\int_{-R}^{R} \phi(s) H_n(x - s) ds \to (4\pi t)^{-1/2} \int_{-R}^{R} \phi(s) \exp(-(x - s)^2/4t) ds.$$

Furthermore, the lim sup of the expression in (6) and (7) are each $\leqq \epsilon$. Thus

$$0 \leqq \limsup_{n \to \infty} \left| \int \phi(s) \left[H_n(x - s) - (4\pi t)^{-1/2} \exp(-(x - s)^2/4t) \right] ds \right| \leqq 2\epsilon.$$

Since ϵ is arbitrary, this proves the lemma.

We complete the proof of the theorem with

LEMMA 7.

$$\lim_{t \to 0+} (4\pi t)^{-1/2} \int \phi(s) \exp(-(x - s)^2/4t) ds = \phi(x)$$

almost everywhere.

This is Corollary 7.2b of Theorem 7.2 of [2, p. 189].

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