A REAL INVERSION FORMULA FOR A CLASS OF CONVOLUTION TRANSFORMS

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In this paper an inversion formula involving only values of \( f(x) \) along the real axis is obtained for the class of convolution transforms described below. A real inversion formula involving \( f(x) \) and its derivatives was obtained by H. Pollard [1] and a complex inversion formula by Hirschman and Widder [2].

We shall prove the following

**Theorem.** Let

\[
(1) \quad f(x) = \int G(x - t)\phi(t)dt,
\]

where

\[
\int G(t) \exp(-ist)dt = \left[ \prod_{k=1}^{\infty} \left( 1 + \frac{s^2}{a_k^2} \right) \right]^{-1} = F(is)
\]

and

\[
\sum_{k=1}^{\infty} \frac{1}{a_k} < \infty, \quad 0 < a_k \leq a_{k+1}, \quad k = 1, 2, \ldots.
\]

If the integral (1) converges for a single real \( x \), it converges for all real \( x \) and is inverted by

\[
\lim_{t \to 0^+} \lim_{n \to \infty} \int f(\xi)d\xi \int \prod_{k=1}^{n} \left( 1 + \frac{u^2}{a_k^2} \right) \cdot \exp[-tu^2 + iu(x - \xi)]du = \phi(x)
\]

for almost every \( x \).

It is assumed that \( \phi(t) \) is Lebesgue integrable on every finite interval and the integral in (1) is interpreted as

\[
\lim_{R \to \infty, \ S \to \infty} \int_{-R}^{R} \int_{-S}^{S}.
\]

When limits are omitted from an integral appearing in the text, the

Presented to the Society, April 22, 1961; received by the editors March 6, 1961 and, in revised form, May 1, 1961.

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range of integration is understood to be \((-\infty, \infty)\).

It is convenient to decompose the proof of the theorem into several lemmas.

**Lemma 1.** If the transform (1) converges for a single real \(x\), it converges for all real \(x\).

This was established in [1]. We note that \(G(t)\) is a class I kernel in the sense of Widder and Hirschman [2, p. 120].

**Lemma 2.**

\[
(2\pi)^{-1} \int F(is) \exp its \, ds
\]

exists and equals \(G(t)\).

By [2, p. 52], \(F(is) = O(|s|^{-p})\) for every \(p\), hence is in \(L(-\infty, \infty)\) from which the result follows immediately.

**Lemma 3.**

\[
\int f(\xi) d\xi \int \prod_{k=1}^{n} \left(1 + \frac{u^2}{a_k^2}\right) \exp \left(-tu^2 - iu(x - \xi)\right) du
\]

exists.

\(f(\xi) = o(\exp(a_1|\xi|))\) by [2, Theorem 2.1, p. 147]. The inner integral is easily shown by a direct evaluation to be \(O(\exp(-A(x-\xi)^2))\), \(0 < A < (4\epsilon)^{-1}\), which establishes the result.

**Lemma 4.**

\[
f_A(\xi) = \int_{-A}^{A} G(\xi - s)\phi(s) ds = O(\exp(a_1|\xi|))
\]

independent of \(A\).

From [2, p. 123] \(G(x-t)/G(-t)\) is nondecreasing or nonincreasing as a function of \(t\) according as \(x\) is greater or less than zero. The asymptotic estimates furnished by [2, Theorem 2.1, p. 108] show furthermore that

\[
\lim_{A \to \infty} G(x - A)/G(-A) = \exp a_1x,
\]
\[
\lim_{A \to -\infty} G(x - A)/G(-A) = \exp(-a_1x).
\]

In the case \(x > 0\), the mean value theorem enables us to write:
\[ f_A(x) = \int_{-A}^{A} [G(x - t)/G(-t)]G(-t)\phi(t)dt \]
\[ = G(x - A)/G(-A) \int_{x}^{A} G(-t)\phi(t)dt, \]
hence
\[ |f_A(x)| \leq B \exp a_1x, \]
where
\[ B = \sup_{a \in R} \int_{R}^{S} G(-t)\phi(t)dt. \]

A similar argument applies in the case \(x < 0\).

**Lemma 5.**

\[ \int f(\xi)d\xi \int \prod_{k=1}^{n} (1 + u^2/a_k^2) \exp (iu(x + \xi) - tu^2)du \]
\[ = \int \phi(s)ds \int \prod_{k=n+1}^{\infty} \{1 + u^2/a_k^2\}^{-1} \exp (iu(x - s) - tu^2)du. \]

The technique employed in the proof of this lemma is the one employed by Blackman [4] in his treatment of convolutions with rational kernels. We have by the definition of \(f(x)\), (3) is equal to

\[ \int d\xi \left[ \lim_{A \to \infty} \int_{-A}^{A} G(\xi - s)\phi(s)ds \right] G_n(x - \xi), \]

where \(G_n(x - \xi)\) is the value of the inner integral in (3). The estimate of Lemma 4 enables us to take the limit outside the outer integral whence (4) becomes

\[ \lim_{A \to \infty} \int G_n(x - \xi)d\xi \int_{-A}^{A} G(\xi - s)\phi(s)ds. \]

By Fubini's theorem, we can interchange the order of integration which replaces (5) by

\[ \lim_{A \to \infty} \int_{-A}^{A} \phi(s)ds \int G_n(x - \xi)G(\xi - s)d\xi. \]

A straightforward calculation shows the inner integral above can be expressed as
\[
\int \left[ \prod_{k=n+1}^{\infty} \left(1 + \frac{u^2}{a_k^2}\right) \right]^{-1} \exp \left( iu(x - s) - tu \right) du,
\]
completing the proof of the lemma.

**Lemma 6.**

\[
\lim_{n \to \infty} \int \phi(s) \left[ H_n(x - s) - \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{(x - s)^2}{4t} \right) \right] ds = 0,
\]

where

\[
H_n(x) = \int \left[ \prod_{k=n+1}^{\infty} \left(1 + \frac{u^2}{a_k^2}\right) \right]^{-1} \exp \left( iux - tu \right) du.
\]

\[
\int \phi(s) \exp \left( -\frac{(x - s)^2}{4t} \right) ds
\]
a exists for every real \( x \). Given \( \epsilon > 0 \), we can for any fixed \( x \) choose \( R \) so large that

\[
\sup_{R < s < \infty} \left( \frac{1}{\sqrt{4\pi t}} \right) \left| \int_{-\infty}^{s} \phi(s) \exp \left( -\frac{(x - s)^2}{4t} \right) ds \right| < \epsilon,
\]

\[
\sup_{R < s < \infty} \left( \frac{1}{\sqrt{4\pi t}} \right) \left| \int_{s}^{-\infty} \phi(s) \exp \left( -\frac{(x - s)^2}{4t} \right) ds \right| < \epsilon.
\]

It is easily established that:

\[
d/dx(H_n(x)/\exp(-x^2/4t)) \geq 0, \quad x \geq 0,
\]

\[
\leq 0, \quad x \leq 0.
\]

By the mean value theorem, if \( R > x \),

\[
\left| \int_{-\infty}^{\infty} \phi(s) H_n(x - s) ds \right|
\]

\[
\leq \left[ H_n(x - R)/\exp(-x R^2/4t) \right] \times \left| \int_{-\infty}^{\epsilon} \phi(s) \exp(-x R^2/4t) ds \right|
\]

\[
\leq \left[ H_n(x + R)/\exp(-(x + R)^2/4t) \right] \times \left| \int_{\epsilon}^{-R} \phi(s) \exp(-(x - s)^2/4t) ds \right|
\]

\[
H_n(x) \text{ tends boundedly to } (4\pi t)^{-1/2} \exp(-x^2/4t), \text{ enabling us to conclude that as } n \to \infty
\]
\[
\int_{-R}^{R} \phi(s)H_n(x - s)ds \to (4\pi t)^{-1/2} \int_{-R}^{R} \phi(s) \exp(-(x - s)^2/4t)ds.
\]

Furthermore, the lim sup of the expression in (6) and (7) are each \( \leq \epsilon \). Thus

\[
0 \leq \limsup_{n \to \infty} \left| \int \phi(s) \left[ H_n(x - s) - (4\pi t)^{-1/2} \exp(-(x - s)^2/4t) \right] ds \right| \leq 2\epsilon.
\]

Since \( \epsilon \) is arbitrary, this proves the lemma.

We complete the proof of the theorem with

**Lemma 7.**

\[
\lim_{t \to 0^+} (4\pi t)^{-1/2} \int \phi(s) \exp(-(x - s)^2/4t)ds = \phi(x)
\]

almost everywhere.

This is Corollary 7.2b of Theorem 7.2 of [2, p. 189].

**Bibliography**


IBM Corporation and

North Carolina State College