A DECOMPOSITION THEOREM FOR
n-DIMENSIONAL MANIFOLDS

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Throughout our discussion an n-dimensional manifold will mean a connected, separable metric space in which each point has an open n-cell neighborhood. Our main result can be stated in the following manner.

**Theorem 1.** Let $M^n$ be an n-dimensional manifold. Then $M^n = P^n \cup C$, where $P^n$ is homeomorphic to euclidean n-space, $E^n$, and $C$ is a closed subset of $M^n$ of dimension at most $n-1$; and $P^n \cap C = \emptyset$.

Considered from one point of view Theorem 1 is a generalization of Corollary 1 in [3]. From still another the result says that any n-manifold is “almost triangulable.” The proof of Theorem 1 leads to more interesting results in the case of compact manifolds which we shall consider presently.

The steps in the proof will be described here. If $C^n$ is a closed n-cell in $M^n$ such that $\text{Bd } C^n$, the boundary of $C^n$, is bicollared in $M^n$, [2], and if $\{a_i\}$ is a countable dense subset of $M^n \setminus C^n$, consider the set $C^n \cup a_i$. Does this set lie on the interior of an n-cell in $M^n$ with a bicollared boundary? If this were the case and if $C_1$ is such an n-cell, one could ask if $C_1 \cup a_2$ lies interior to an n-cell in $M^n$ with a bicollared boundary. Continuing in this way with sets of the form $C_i \cup a_{i+1}$, if such enclosure is always possible, we obtain an increasing sequence $\{C_i\}$ of closed n-cells in $M^n$, $\text{Bd } C_i$ is bicollared in $M^n$ and $\text{int } C_{i+1} \supseteq C_i$, where interior $C_{i+1}$ is written $\text{int } C_i$. Next we observe that $P^n = \bigcup_i C_i$ is $E^n$ by either a direct construction of cells with annuli between them or by applying the main result of [1]. Then $M^n - P^n = C$ is nowhere dense in $M^n$ and closed since $P^n$ is open. The sets $P^n$ and $C$ would then meet the requirements of Theorem 1.

From this outline it is clear that the proof of Theorem 1 follows immediately from a lemma.

**Lemma 1.** Let $M^n$ be an n-manifold and $D^n$ a closed n-cell in $M^n$ with bicollared boundary. Then if $p$ is any point in $M^n$, $D^n \cup p$ lies in $\text{int } D_i^n$, where $D_i^n$ is a closed n-cell and $\text{Bd } D_i^n$ is bicollared.

**Proof.** Let $q$ be any point in $\text{int } D^n$. There is a homeomorphism $h$ of $M^n$ onto $M^n$ which is pointwise fixed outside any neighborhood $V$ of $D^n$ and which carries $D^n$ into any preassigned neighborhood $U$ of $q$.
while \( h(q) = q \). This follows from the fact that \( \text{Bd} \, D^n \) is bicciliated.

Since \( M^n \) is a manifold, the set \( p \cup q \) lies interior to an \( n \)-cell in \( M^n \) and so interior to an \( n \)-cell with a bicciliated boundary in \( M^n \). Evidently we need only select \( \mathcal{U} \) in the interior of such a cell to obtain the proof of Lemma 1.

With the proof of Lemma 1 we obtain Theorem 1. In the case \( M^n \) is compact the conditions on \( C \) are stronger.

**Theorem 2.** Let \( M^n \) be a compact \( n \)-dimensional manifold. Then \( M^n = P^n \cup C \), where \( P^n \) is homeomorphic to \( E^n \), and \( C \) is a nonseparating continuum in \( M^n \); \( P^n \cap C = \emptyset \).

It is convenient to call the decomposition \( P^n \cup C \) of \( M^n \) in the above theorems a standard decomposition if \( P^n \) is obtained as in the proof of Theorem 1.

**Corollary 1.** Let \( M^n \) be a compact \( n \)-manifold and \( M^n = P^n \cup C \) a standard decomposition of \( M^n \). Then if there is a homeomorphism \( h \) of \( M^n \) onto \( M^n \) such that \( h(C) \subset P^n \), then \( M^n \) is an \( n \)-sphere.

**Proof.** By Theorem 2, \( C \) is compact and so \( h(C) \) lies in the interior of a closed \( n \)-cell \( C^* \) in \( P^n \). By the construction of \( P^n \), \( M^n \) is the union of two closed \( n \)-cells with no boundary points in common. Whence, as in Lemma 3 of [4], one can conclude that \( M^n \) is a sphere.

**Corollary 2.** Let \( M^n \) be a compact \( n \)-compact \( n \)-manifold and let \( M^n = P^n_1 \cup C_1 = P^n_2 \cup C_2 \) be two standard decompositions. If \( C_1 \cap C_2 = \emptyset \), then \( M^n \) is a sphere.

It should be pointed out that the set \( C \) in a standard decomposition need not be nice. In the case of the 2-sphere \( S^2 \), \( C \) may be any nonseparating 1-dimensional continuum in \( S^2 \); so \( C \) need not be locally connected.

**Theorem 3.** Let \( M^n \) be an \( n \)-manifold and \( S^n \), the \( n \)-sphere. Then there is a map \( f \) from \( M^n \) onto \( S^n \) such that each point of \( S^n \) has a degenerate inverse except perhaps for one point \( p \), and \( \dim f^{-1}(p) \leq n - 1 \).

**Proof.** The representation of \( P^n \) as an increasing sequence of \( n \)-cells provides an evident map of the type described with \( C \) as the only possible nondegenerate inverse. In case \( M^n = E^n \), \( C \) may be void; however, one may arrange it so that \( C \) is not void even in this case.

In the proof of Corollary 1 to Theorem 2 we observed that a compact \( n \)-manifold which fails to be a sphere cannot be the union of two closed \( n \)-cells having no boundary points in common. Similar results
may be obtained for open regions. If $M^n$ is a compact $n$-manifold, $P^n \cup C = M^n$, a standard decomposition, let $C$ be in an open set $U$ in $M^n$ such that $U$ is homeomorphic to a subset of $E^n$. Then if $h$ is an imbedding of $U$ in $E^n$ we note that $h(C)$ is the limit in $E^n$ of a strictly decreasing sequence of closed $n$-cells with bicollared boundaries in $h(U) \subset E^n$. Whence, we obtain $M^n$ as a union of closed $n$-cells with disjoint bicollared boundaries and so $M^n$ is an $n$-sphere. We can then assert another theorem.

**Theorem 4.** If $M^n$ is a compact $n$-manifold which is not an $n$-sphere and if $M^n = P^n \cup C$ is a standard decomposition, then $C$ has no neighborhood in $M^n$ which can be imbedded in $E^n$.

**References**


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