A DECOMPOSITION THEOREM FOR n-DIMENSIONAL MANIFOLDS

P. H. DOYLE AND J. G. HOCKING

Throughout our discussion an n-dimensional manifold will mean a connected, separable metric space in which each point has an open n-cell neighborhood. Our main result can be stated in the following manner.

THEOREM 1. *Let M^n be an n-dimensional manifold. Then $M^n = P^n \cup C$, where P^n is homeomorphic to euclidean n-space, E^n, and C is a closed subset of M^n of dimension at most $n-1$; and $P^n \cap C = \square$.*

Considered from one point of view Theorem 1 is a generalization of Corollary 1 in [3]. From still another the result says that any n-manifold is "almost triangulable." The proof of Theorem 1 leads to more interesting results in the case of compact manifolds which we shall consider presently.

The steps in the proof will be described here. If C^n is a closed n-cell in M^n such that Bd C^n, the boundary of C^n, is bicollared in M^n, [2], and if $\{a_i\}$ is a countable dense subset of $M^n \setminus C^n$, consider the set $C^n \cup a_1$. Does this set lie on the interior of an n-cell in M^n with a bicollared boundary? If this were the case and if C_1 is such an n-cell, one could ask if $C_1 \cup a_2$ lies interior to an n-cell in M^n with a bicollared boundary. Continuing in this way with sets of the form $C_i \cup a_{i+1}$, if such enclosure is always possible, we obtain an increasing sequence $\{C_i\}$ of closed n-cells in M^n, Bd C_i is bicollared in M^n and int $C_{i+1} \supset C_i$, where interior C_{i+1} is written int C_{i+1}. Next we observe that $P^n = \bigcup_i C_i$ is E^n by either a direct construction of cells with annuli between them or by applying the main result of [1]. Then $M^n - P^n = C$ is nowhere dense in M^n and closed since P^n is open. The sets P^n and C would then meet the requirements of Theorem 1.

From this outline it is clear that the proof of Theorem 1 follows immediately from a lemma.

LEMMA 1. *Let M^n be an n-manifold and D^n a closed n-cell in M^n with bicollared boundary. Then if p is any point in M^n, $D^n \cup p$ lies in* int D_1^n, *where D_1^n is a closed n-cell and* Bd D_1^n *is bicollared.*

PROOF. Let q be any point in int D^n. There is a homeomorphism h of M^n onto M^n which is pointwise fixed outside any neighborhood V of D^n and which carries D^n into any preassigned neighborhood $\mathfrak{U}$ of q

Received by the editors May 11, 1961.

while $h(q)=q$. This follows from the fact that Bd D^n is bicollared.

Since M^n is a manifold, the set $p\cup q$ lies interior to an n-cell in M^n and so interior to an n-cell with a bicollared boundary in M^n. Evidently we need only select $\mathfrak{U}$ in the interior of such a cell to obtain the proof of Lemma 1.

With the proof of Lemma 1 we obtain Theorem 1. In the case M^n is compact the conditions on C are stronger.

THEOREM 2. *Let M^n be a compact n-dimensional manifold. Then $M^n=P^n\cup C$, where P^n is homeomorphic to E^n, and C is a nonseparating continuum in M^n; $P^n\cap C=\square$.*

It is convenient to call the decomposition $P^n\cup C$ of M^n in the above theorems a standard decomposition if P^n is obtained as in the proof of Theorem 1.

COROLLARY 1. *Let M^n be a compact n-manifold and $M^n=P^n\cup C$ a standard decomposition of M^n. Then if there is a homeomorphism h of M^n onto M^n such that $h(C)\subset P^n$, then M^n is an n-sphere.*

PROOF. By Theorem 2, C is compact and so $h(C)$ lies in the interior of a closed n-cell C^n in P^n. By the construction of P^n, M^n is the union of two closed n-cells with no boundary points in common. Whence, as in Lemma 3 of [4], one can conclude that M^n is a sphere.

COROLLARY 2. *Let M^n be a compact n-compact n-manifold and let $M^n=P_1^n\cup C_1=P_2^n\cup C_2$ be two standard decompositions. If $C_1\cap C_2=\square$, then M^n is a sphere.*

It should be pointed out that the set C in a standard decomposition need not be nice. In the case of the 2-sphere S^2, C may be any nonseparating 1-dimensional continuum in S^2; so C need not be locally connected.

THEOREM 3. *Let M^n be an n-manifold and S^n, the n-sphere. Then there is a map f from M^n onto S^n such that each point of S^n has a degenerate inverse except perhaps for one point p, and* $\dim f^{-1}(p)\leq n-1$.

PROOF. The representation of P^n as an increasing sequence of n-cells provides an evident map of the type described with C as the only possible nondegenerate inverse. In case $M^n=E^n$, C may be void; however, one may arrange it so that C is not void even in this case.

In the proof of Corollary 1 to Theorem 2 we observed that a compact n-manifold which fails to be a sphere cannot be the union of two closed n-cells having no boundary points in common. Similar results may be obtained for open regions. If M^n is a compact n-manifold, $P^n \cup C = M^n$, a standard decomposition, let C be in an open set $\mathcal{U}$ in M^n such that $\mathcal{U}$ is homeomorphic to a subset of E^n. Then if h is an imbedding of $\mathcal{U}$ in E^n we note that $h(C)$ is the limit in E^n of a strictly decreasing sequence of closed n-cells with bicollared boundaries in $h(\mathcal{U}) \subset E^n$. Whence, we obtain M^n as a union of closed n-cells with disjoint bicollared boundaries and so M^n is an n-sphere. We can then assert another theorem.

THEOREM 4. *If M^n is a compact n-manifold which is not an n-sphere and if $M^n = P^n \cup C$ is a standard decomposition, then C has no neighborhood in M^n which can be imbedded in E^n.*

REFERENCES

1. M. Brown, *The monotone union of open n-cells is an open n-cell*, Proc. Amer. Math. Soc. **12** (1961), 812–814.

2. M. Brown and E. Michael, *Collared subsets of metric spaces*, Abstract 576-128, Notices Amer. Math. Soc. **7** (1960), 940.

3. M. K. Fort, Jr., *A theorem about topological n-cells*, Proc. Amer. Math. Soc. **5** (1954), 456–459.

4. J. R. Stallings, *Polyhedral homotopy-spheres*, Bull. Amer. Math. Soc. **66** (1960), 485–488.

MICHIGAN STATE UNIVERSITY