

THE TORSION PONTRYAGIN CLASSES

EMERY THOMAS¹

Let η be a real n -plane bundle over a paracompact space B . With η one associates two (total) characteristic classes: the Stiefel-Whitney class

$$W(\eta) = 1 + W_1(\eta) + W_2(\eta) + \cdots + W_n(\eta) \quad (W_i(\eta) \in H^i(B; Z_2))$$

and the Pontryagin class

$$p(\eta) = 1 + p_1(\eta) + \cdots + p_q(\eta) \quad (p_i(\eta) \in H^{4i}(B; Z)).$$

Here $Z = \text{integers}$, $Z_2 = \text{integers mod } 2$, and $q = [n/2]$ —the largest integer in $n/2$.

Recall that the homogeneous Pontryagin classes, $p_i(\eta)$, are defined by

$$p_i(\eta) = (-1)^i c_{2i}(\eta_c) \quad (1 \leq i \leq [n/2]),$$

where η_c is the complexification of the bundle η and $c_j(\eta_c)$ denotes the j th Chern class of the complex n -plane bundle η_c (see [1]).

For convenience of notation set

$$P_{2k}(\eta) = (-1)^q c_k(\eta_c),$$

where $q = [k/2]$ and $1 \leq k \leq 2n$. Thus, $p_i(\eta) = P_{4i}(\eta)$. We call the classes $P_{4k+2}(\eta)$ ($k \geq 0$) the *torsion Pontryagin classes* of η , since [1] $2P_{4k+2}(\eta) = 0$. These torsion classes occur in an important situation—that is, in the expression for the Pontryagin classes of the (Whitney) sum of two bundles. The purpose of this note is to express these torsion classes in terms of the standard characteristic classes of the bundle η . In fact, we give three equivalent expressions for them!

THEOREM. *Let η be a real n -plane bundle. Then, for $k \geq 0$,*

$$\begin{aligned} P_{4k+2}(\eta) &= \delta_*(W_{2k}(\eta)W_{2k+1}(\eta)) = p_k(\eta)\delta_*W_1(\eta) + (\delta_*W_{2k}(\eta))^2 \\ &= \delta_*Sq^{2k}W_{2k+1}(\eta). \end{aligned}$$

Here δ_* is the Bockstein coboundary associated with the exact sequence

$$0 \rightarrow Z \xrightarrow{2} Z \rightarrow Z_2 \rightarrow 0,$$

and Sq^{2k} is the Steenrod cohomology operator.

Received by the editors April 25, 1961.

¹ Research supported in part by U. S. Air Force contract 49(638)-79.

By the usual naturality argument it suffices to prove the theorem for the case $\eta = \gamma^n$, the classifying bundle over the real Grassmann complex G_n —that is, the space whose points are all n -dimensional linear subspaces of R^∞ (R = real numbers; see [1] for details).

Denote by P^∞ the real, infinite-dimensional projective space ($P^\infty = G_1$) and by γ_i^1 ($1 \leq i \leq n$) the 1-plane bundle over the n -fold Cartesian product $(P^\infty)^n$ induced from γ^1 by the i th projection map $(P^\infty)^n \rightarrow P^\infty$. Set

$$\xi^n = \gamma_1^1 \oplus \cdots \oplus \gamma_n^1 \quad (\oplus = \text{Whitney sum}).$$

By the same argument as that used to prove Theorem 5.1 of [2], one obtains

$$\rho_2 P_{4k+2}(\xi^n) = W_{2k+1}(\xi^n)^2,$$

where ρ_r is induced by the factor homomorphism $Z \rightarrow Z_r$ ($r = 2, 3, \dots$).

Let f be a map from $(P^\infty)^n$ to G_n which induces ξ^n from γ^n . Since f_2^* , the mod 2 cohomology homomorphism induced by f , is a monomorphism we obtain—by naturality of the characteristic classes,

$$(1) \quad \rho_2 P_{4k+2}(\gamma^n) = W_{2k+1}(\gamma^n)^2.$$

Denote by T the torsion subgroup of $H^*(G_n; Z)$. Since every element in T has order 2 it follows that two classes, x and y , in T are equal if, and only if, $\rho_2(x) = \rho_2(y)$. Now, as remarked above, $P_{4k+2}(\gamma^n) \in T$. Therefore, the first equality in our theorem follows from (1) when we note that

$$\begin{aligned} \rho_2 \delta_* (W_{2k} W_{2k+1}) &= \beta_2 (W_{2k} W_{2k+1}) \\ &= (W_{2k+1} + W_1 W_{2k})(W_{2k+1}) + W_{2k}(W_1 W_{2k+1}) \\ &= W_{2k+1}^2. \end{aligned}$$

Here $W_i = W_i(\gamma^n)$, and β_2 is the derivation $\rho_2 \delta_*$ which has the values

$$\begin{aligned} \beta_2 W_{2i}(\eta) &= W_1(\eta) W_{2i}(\eta) + W_{2i+1}(\eta), \\ \beta_2 W_{2i-1}(\eta) &= W_1(\eta) W_{2i-1}(\eta) \end{aligned} \quad (i \geq 1),$$

for any bundle η .

The second equality in the theorem follows likewise from (1):

$$\begin{aligned} \rho_2 (p_k \delta_* W_1 + (\delta_* W_{2k})^2) &= (\rho_2 p_k)(\rho_2 \delta_* W_1) + (\rho_2 \delta_* W_{2k})^2 \\ &= (\rho_2 p_k)(\beta_2 W_1) + (\beta_2 W_{2k})^2 \\ &= (W_{2k})^2 (W_1)^2 + (W_1 W_{2k} + W_{2k+1})^2 \\ &= (W_{2k+1})^2 = \rho_2 P_{4k+2}. \end{aligned}$$

Here we have used the fact that ρ_2 is a multiplicative homomorphism and that $\rho_2 p_k = W_{2k}^2 (p_k = p_k(\gamma^n))$.

Finally, to obtain the third equality notice that

$$\rho_2 \delta_* S q^{2k} = \beta_2 S q^{2k} = S q^1 S q^{2k} = S q^{2k+1}.$$

Since W_{2k+1} is a $(2k+1)$ -dimensional cohomology class,

$$S q^{2k+1} W_{2k+1} = (W_{2k+1})^2,$$

and therefore

$$\rho_2 \delta_* S q^{2k} W_{2k+1} = (W_{2k+1})^2 = \rho_2 P_{4k+2},$$

completing the proof of the theorem.

Suppose now that η and ζ are two bundles over the same base space. Define for $k \geq 1$,

$$D_k(\eta, \zeta) = p_k(\eta \oplus \zeta) - \sum_{i+j=k} p_i(\eta) p_j(\zeta);$$

that is, D_k measures the deviation of the Pontryagin class p_k from following the multiplicative rule for characteristic classes. Using the fact that $(\eta \oplus \zeta)_e$ is equivalent to $\eta_e \oplus \zeta_e$, together with the multiplicative property of the Chern class, one has

$$P_{4k}(\eta \oplus \zeta) = \sum_{i+j=2k} P_{2i}(\eta) P_{2j}(\zeta).$$

Thus, using the first equality in the theorem, we obtain:²

(2) *Let η, ζ be bundles over the same base space. Then, for $k \geq 1$,*

$$D_k(\eta, \zeta) = \sum_{i+j=k-1} \delta_* [W_{2i}(\eta) W_{2i+1}(\eta)] \delta_* [W_{2j}(\zeta) W_{2j+1}(\zeta)].$$

Suppose that η and ζ are both orientable bundles. Then, $W_1(\eta) = W_1(\zeta) = 0$, and consequently,

$$D_1(\eta, \zeta) = D_2(\eta, \zeta) = 0.$$

Continuing in this fashion, recall that an n -plane bundle ξ is equivalent to a bundle with $\text{Spin}(n)$ as structure group if, and only if, $W_1(\xi) = W_2(\xi) = 0$. Denote by $B_{\text{Spin}(n)}$ a classifying space for $\text{Spin}(n)$ -bundles. Then one easily shows that

$$H^i(B_{\text{Spin}(n)}; Z) = 0, \quad 1 \leq i \leq 3; \quad H^4(B_{\text{Spin}(n)}; Z) \text{ is free abelian.}$$

Thus for a $\text{Spin}(n)$ -bundle ξ ,

² Cartan notes an equivalent formula in Exposé 17 (equation no. 59') of Séminaire H. Cartan 1959/1960, Paris.

$$W_1(\xi) = W_3(\xi) = W_5(\xi) = 0.$$

Applying this information to equation (2) we obtain:

(3) Let η, ζ be orientable bundles over the same base and suppose that

$$W_2(\eta) = W_2(\zeta) = 0.$$

Then,

$$D_k(\eta, \zeta) = 0, \quad \text{for } 1 \leq k \leq 6.$$

There is one other place where the torsion Pontryagin classes play a role. Denote by \mathfrak{P}_2 the Pontryagin square cohomology operation and by θ_2 the cohomology homomorphism induced by the inclusion $Z_2 \rightarrow Z_4$. Then, Wu has shown (see [2; 3]) that

$$\mathfrak{P}_2(W_{2i+1}(\eta)) = \rho_4 \delta_* Sq^{2i} W_{2i+1}(\eta) + \theta_2(W_1(\eta) Sq^{2i} W_{2i+1}(\eta)),$$

where η is any real n -plane bundle. Thus, by the third equality of our theorem we obtain:

(4) Let η be an n -plane bundle. Then, for $i \geq 0$,

$$\mathfrak{P}_2(W_{2i+1}(\eta)) = \rho_4 P_{4i+2}(\eta) + \theta_2(W_1(\eta) Sq^{2i} W_{2i+1}(\eta)).$$

BIBLIOGRAPHY

1. J. Milnor, *Lectures on characteristic classes*, mimeographed notes, Princeton University, 1957.
2. E. Thomas, *On the cohomology of the real Grassmann complexes . . .*, Trans. Amer. Math. Soc. **96** (1960), 67-89.
3. W. Wu, *On Pontryagin classes*. III, Acta Math. Sinica **4** (1954), 323-347.

UNIVERSITY OF CALIFORNIA AT BERKELEY