ON A PAPER OF REICH CONCERNING MINIMAL SLIT DOMAINS

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1. In a recent paper [2] Reich has made the observation that an example of Koebe given in 1918 [1] does not fulfill its asserted purpose. This example was to show that vanishing measure of the complement did not assure that a slit domain was minimal. Reich proceeded to fill the gap by carrying out the following somewhat more general construction.

Let $A$ be a compact perfect nowhere dense set on the $x$-axis in the $z$-plane ($z = x + iy$). Then there exists a compact set $S$ in the $z$-plane with the properties:

(i) $A$ is the projection of $S$ on the $x$-axis,
(ii) $S$ is composed of segments symmetric with respect to the $x$-axis, at least one segment being present,
(iii) any point in $S$, not on the $x$-axis and not at the end of a segment in $S$, is the limit, both from the left and right, of points of $S$.

Once this construction is performed the desired examples are easily given [2, §4]. The object of the present paper is to give an alternative construction which is very explicit and direct.

2. Since any two sets such as $A$ are homeomorphic under a self-homeomorphism of the $x$-axis it is enough to perform the construction for the Cantor ternary set $C$ on the interval $[0, 1]$. This set consists of those points which admit a ternary decimal expansion containing only 0's and 2's. With this restriction the representation is unique. In this representation of $d \in C$ let the initial position of the first block of length $n$ (consisting entirely either of 0's or 2's) for $n$ integral $\geq 2$ be denoted by $N_n(d)$. We define

$$L(d) = \exp \left[ - \sum_{j=2}^{\infty} N_j^{-1}(d) \right]$$

if the series $\sum_{j=2}^{\infty} N_j^{-1}(d)$ converges and

$$L(d) = 0$$

if the series in question diverges. Naturally only those values $N_n(d)$ which are defined appear so that if the length of blocks is bounded
the series terminates, so evidently converges. In particular if there is no block of length 2, \( L(d) = 1 \), if there is an infinite block, \( L(d) = 0 \).

Now we define the set \( S \) to consist of points \((x, y)\) with

\[
x \in \mathbb{C}, \quad -l(x) \leq y \leq l(x).
\]

The desired properties of \( S \) are evident except for compactness and property (iii), both of which we now prove.

To show \( S \) is closed let \( d, d_j \in \mathbb{C}, \lim_{j \to \infty} d_j = d \). If \( L(d) > 0 \), given \( \epsilon > 0 \) let \( N_n(d) \) be defined for \( 2 \leq n \leq T \) and

\[
\sum_{n=2}^{T} N_{-1}(d) > \sum_{n=2}^{\infty} N_{-1}(d) - \epsilon.
\]

As soon as \( j \) is large enough \( N_n(d_j) = N_n(d) \), \( 2 \leq n \leq T \) so \( L(d_j) < \epsilon L(d) \).

If \( L(d) = 0 \) given prescribed \( M \), let

\[
\sum_{n=2}^{T} N_{-1}(d) > M.
\]

Then as before for \( j \) large enough \( L(d_j) < \epsilon^{-M} \).

To prove property (iii) we need consider only \( d \in \mathbb{C} \) with \( L(d) > 0 \). Then in any terminal portion of the decimal expansion of \( d \) there are both 0's and 2's. Given a positive integer \( M \) at positions of index greater than \( M \) there will be an adjacent pair consisting of a 0 and a 2 in either order. Let us choose \( d' \) forming its decimal expansion by choosing an entry 2 beyond this pair and replacing the decimal expansion of \( d \) starting with this by alternate 0's and 2's. Let us choose \( d'' \) forming its decimal expansion by choosing an entry 0 beyond this pair and replacing the decimal expansion of \( d \) starting with this by alternate 2's and 0's. Then whenever \( N_j(d') \) is defined so is \( N_j(d) \) with

\[
N_j(d') \geq N_j(d)
\]

apart from possibly one term with

\[
N_j(d') > M.
\]

Thus

\[
\sum_{j=2}^{\infty} N_{-1}(d') < \sum_{j=2}^{\infty} N_{-1}(d) + M^{-1}
\]

so that

\[
L(d') > \epsilon M^{-1} L(d).
\]

Similar remarks apply to \( d'' \), we have \( d' < d < d'' \) and we can make \( d' \),
as close as we please to $d$. This completes the proof that $S$ has property (iii).

**Bibliography**


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**IDEALS OF SQUARE SUMMABLE POWER SERIES**

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Let $\mathcal{C}$ be a Hilbert space. By $\mathcal{C}(z)$ we mean the Hilbert space of all formal power series $f(z) = \sum a_n z^n$ in the indeterminate $z$ with coefficients $a_n$ in $\mathcal{C}$, such that

$$
\|f\|_2^2 = \sum \|a_n\|^2 < \infty.
$$

Although $\mathcal{C}(z)$ may be thought of as a space of $\mathbb{C}$-valued functions analytic in the unit disc, we prefer the point of view adopted above in which the notation $\mathcal{C}(z)$ is used as the algebraist uses $K[x]$ for the ring of polynomials in $x$ over a field $K$.

An ideal of $\mathcal{C}(z)$ is a vector subspace $\mathfrak{M}$ of $\mathcal{C}(z)$ which contains $zf(z)$ whenever it contains $f(z)$. We will obtain the structure of the closed ideals of $\mathcal{C}(z)$. This problem was solved in [6] when $\mathcal{C}$ has dimension 1, and in this form may be regarded as an interpretation of the work of Beurling [1]. One advantage of our formulation is that it generalizes naturally to Hilbert spaces $\mathcal{C}$ of arbitrary dimension.

The solution is in terms of formal power series $B(z) = \sum B_n z^n$ whose coefficients $B_n$ are bounded linear transformations (i.e., operators) on $\mathcal{C}$. We write $N = N(B)$ for the set of all $\mathfrak{c}$ in $\mathfrak{M}$ such that $B(z)\mathfrak{c} = \sum B_n \mathfrak{c} z^n$ vanishes identically; in other words, $N$ is the intersection of the null spaces of the operators $B_n$. The series $B(z)$ of interest are those which satisfy (1), whenever $(\mathfrak{c}_n)$ is a sequence of unit vectors in $\mathfrak{M}$ orthogonal to $N$, $(\mathfrak{c}_n B(z) \mathfrak{c}_n)$ is an orthonormal set in $\mathcal{C}(z)$. When (1) is satisfied we write $\mathfrak{M}(B)$ for the set of all formal products $B(z)f(z)$ with $f(z)$ in

Received by the editors January 21, 1961 and, in revised form, April 3, 1961.