ON A PAPER OF REICH CONCERNING MINIMAL SLIT DOMAINS[1]

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1. In a recent paper [2] Reich has made the observation that an example of Koebe given in 1918 [1] does not fulfill its asserted purpose. This example was to show that vanishing measure of the complement did not assure that a slit domain was minimal. Reich proceeded to fill the gap by carrying out the following somewhat more general construction.

Let A be a compact perfect nowhere dense set on the x-axis in the z-plane ($z=x+iy$). Then there exists a compact set S in the z-plane with the properties:

(i) A is the projection of S on the x-axis,

(ii) S is composed of segments symmetric with respect to the x-axis and points on the x-axis, at least one segment being present,

(iii) any point in S, not on the x-axis and not at the end of a segment in S, is the limit, both from the left and right, of points of S.

Once this construction is performed the desired examples are easily given [2, §4]. The object of the present paper is to give an alternative construction which is very explicit and direct.

2. Since any two sets such as A are homeomorphic under a self-homeomorphism of the x-axis it is enough to perform the construction for the Cantor ternary set C on the interval $[0, 1]$. This set consists of those points which admit a ternary decimal expansion containing only 0's and 2's. With this restriction the representation is unique. In this representation of $d \in C$ let the initial position of the first block of length n (consisting entirely either of 0's or 2's) for n integral $\geqq 2$ be denoted by $N_n(d)$. We define

$$l(d) = \exp\left[- \sum_{j=2}^{\infty} N_j^{-1}(d) \right]$$

if the series $\sum_{j=2}^{\infty} N_j^{-1}(d)$ converges and

$$l(d) = 0$$

if the series in question diverges. Naturally only those values $N_n(d)$ which are defined appear so that if the length of blocks is bounded

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the series terminates, so evidently converges. In particular if there is no block of length 2, $l(d)=1$, if there is an infinite block, $l(d)=0$.

Now we define the set S to consist of points (x, y) with

$$x \in C, \qquad -l(x) \leqq y \leqq l(x).$$

The desired properties of S are evident except for compactness and property (iii), both of which we now prove.

To show S is closed let d, $d_j \in C$, $\lim_{j\to\infty} d_j = d$. If $l(d)>0$, given $\epsilon>0$ let $N_n(d)$ be defined for $2 \leqq n \leqq T$ and

$$\sum_{n=2}^{T} N_n^{-1}(d) > \sum_{n=2}^{\infty} N_n^{-1}(d) - \epsilon.$$

As soon as j is large enough $N_n(d_j) = N_n(d)$, $2 \leqq n \leqq T$ so $l(d_j) < e^{\epsilon} l(d)$. If $l(d)=0$ given prescribed M, let

$$\sum_{n=2}^{T} N_n^{-1}(d) > M.$$

Then as before for j large enough $l(d_j) < e^{-M}$.

To prove property (iii) we need consider only $d \in C$ with $l(d)>0$. Then in any terminal portion of the decimal expansion of d there are both 0's and 2's. Given a positive integer M at positions of index greater than M there will be an adjacent pair consisting of a 0 and a 2 in either order. Let us choose d' forming its decimal expansion by choosing an entry 2 beyond this pair and replacing the decimal expansion of d starting with this by alternate 0's and 2's. Let us choose d'' forming its decimal expansion by choosing an entry 0 beyond this pair and replacing the decimal expansion of d starting with this by alternate 2's and 0's. Then whenever $N_j(d')$ is defined so is $N_j(d)$ with

$$N_j(d') \geqq N_j(d)$$

apart from possibly one term with

$$N_j(d') > M.$$

Thus

$$\sum_{j=2}^{\infty} N_j^{-1}(d') < \sum_{j=2}^{\infty} N_j^{-1}(d) + M^{-1}$$

so that

$$l(d') > e^{-M^{-1}} l(d).$$

Similar remarks apply to d'', we have $d' < d < d''$ and we can make d',

d'' as close as we please to d. This completes the proof that S has property (iii).

Bibliography

1. P. Koebe, *Zur konformen Abbildung unendlich-vielfach zusammenhängender schlichter Bereiche auf Schlitzbereiche*, Nachr. Kgl. Ges. Wiss. Göttingen Math.-Phys. Kl. (1918), 60–71.

2. E. Reich, *A counterexample of Koebe's for slit mappings*, Proc. Amer. Math. Soc. **11** (1960), 970–975.

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IDEALS OF SQUARE SUMMABLE POWER SERIES

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Let $\mathcal{C}$ be a Hilbert space. By $\mathcal{C}(z)$ we mean the Hilbert space of all formal power series $f(z) = \sum a_n z^n$ in the indeterminate z with coefficients a_n in $\mathcal{C}$, such that

$$\|f\|^2 = \sum \|a_n\|^2 < \infty. \tag{1}$$

Although $\mathcal{C}(z)$ may be thought of as a space of $\mathcal{C}$-valued functions analytic in the unit disc, we prefer the point of view adopted above in which the notation $\mathcal{C}(z)$ is used as the algebraist uses $K[x]$ for the ring of polynomials in x over a field K.

An ideal of $\mathcal{C}(z)$ is a vector subspace $\mathfrak{M}$ of $\mathcal{C}(z)$ which contains $zf(z)$ whenever it contains $f(z)$. We will obtain the structure of the closed ideals of $\mathcal{C}(z)$. This problem was solved in [6] when $\mathcal{C}$ has dimension 1, and in this form may be regarded as an interpretation of the work of Beurling [1]. One advantage of our formulation is that it generalizes naturally to Hilbert spaces $\mathcal{C}$ of arbitrary dimension.

The solution is in terms of formal power series $B(z) = \sum B_n z^n$ whose coefficients B_n are bounded linear transformations (i.e., operators) on $\mathcal{C}$. We write $N = N(B)$ for the set of all c in $\mathcal{C}$ such that $B(z)c = \sum B_n c z^n$ vanishes identically; in other words, N is the intersection of the null spaces of the operators B_n. The series $B(z)$ of interest are those which satisfy (1), whenever (c_n) is a sequence of unit vectors in $\mathcal{C}$ orthogonal to N, $(z^n B(z) c_n)$ is an orthonormal set in $\mathcal{C}(z)$. When (1) is satisfied we write $\mathfrak{M}(B)$ for the set of all formal products $B(z)f(z)$ with $f(z)$ in

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