ON TRANSVERSALS OF SIMPLY CONNECTED REGIONS

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1. **Introduction.** In connection with work on the boundary behavior under conformal mapping [4] I have had to consider certain curves, called "transversals"[2] below. These are generalizations of crosscuts of a simply connected region of the extended plane. Transversals are defined by the principal separation property of crosscuts; namely, they divide a simply connected region into two regions that are again simply connected. This note is devoted to proving the theorem that the union of two transversals of a simply connected region has a complement (relative to the region) all of whose components are themselves simply connected regions. Furthermore, the relative boundary of each of these regions is either a Jordan curve or a union of transversals.

Our theorem is an extension of a theorem of Kerékjártó [2, p. 87; 3, p. 168; 5, p. 108] to the effect that all complementary regions of a pair of intersecting Jordan curves are simply connected and have Jordan curves for their boundaries. The proof is based on the observation that our theorem is actually a version of Kerékjártó's result, provided the latter is applied on the Alexandroff one-point compactification (see for example [1, p. 23]) of the given region.

Transversals have the desirable property that they are invariant under topological mappings of the (open) region. This is no longer true of ordinary crosscuts.

2. **Definitions.** We recall that a *region* is an open connected subset of the extended plane. A region is called *simply connected* in case its boundary is a continuum (which may degenerate to a point) or empty, that is, in case its complement in the extended plane is connected.

A *crosscut* of a region Ω is a homeomorphic image of the open interval $(0, 1)$ in Ω with the property that the homeomorphism is the restriction of a continuous map of the closed interval $[0, 1]$ into the closure $\bar{\Omega}$ of Ω, where the images of 0 and 1 lie on $\Gamma = \text{bd } \Omega$. (We do not exclude the possibility that these two "endpoints" of the crosscut

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[2] These curves were actually called "crosscuts" in [4]. The theorem of the present paper enables one to prove Theorem 4 of [4] without the intervention of "generalized crosscuts"—certain unions of transversals.

coincide.) A crosscut of a simply connected region Ω separates Ω into two simply connected regions (see, for example, [**2**, p. 106; **5**, p. 110]).

A *transversal* γ of a simply connected region Ω is a homeomorphic image of the open unit interval (0, 1) in Ω with the property that the set-theoretic difference $\Omega-\gamma$ has exactly two components and that these are simply connected. Thus, every crosscut of a simply connected region is a transversal of the region. The converse of this statement is false, since a transversal need not have "endpoints."

Let Ω be a region. We denote its one-point compactification by Ω^*: $\Omega^*=\Omega\cup(p_\infty)$, where "$p_\infty$" designates the added compactifying point. If γ is a subset of the region Ω we shall use $\bar{\gamma}$ and γ^* for the closures of γ in the extended plane and in the compactification Ω^*, respectively.

3. **The Theorem.** We state Kerékjártó's theorem and give two auxiliary propositions that are needed to establish the main result.

KERÉKJÁRTÓ'S THEOREM. *Let J_1 and J_2 be Jordan curves of the extended plane, and let $J=J_1\cup J_2$. If J_1 and J_2 have more than one point in common, the boundary of each component of the complement* (*with respect to the extended plane*) *of J is itself a Jordan curve.*

LEMMA. *Let Ω be a region and let Γ be its boundary* (*relative to the extended plane*). *If γ is a subset of Ω then $\bar{\gamma}$ intersects Γ if and only if $p_\infty\in\gamma^*$.*

PROOF. By definition, $p_\infty\in\gamma^*$ if and only if γ meets the complement of every compact subset of Ω. Clearly, this is equivalent with the condition $\bar{\gamma}\cap\Gamma\neq\varnothing$.

COROLLARY. *If γ is a transversal of a simply connected region Ω then γ^* is homeomorphic to a circumference. The homeomorphism can be realized as an extension to* $[0, 1]$ *of the homeomorphism ϕ_0 of $I^0=(0, 1)$ to γ, with the images of* 0 *and of* 1 *identified at p_∞.*

PROOF. It follows from the separating properties of transversals and from the lemma that $p_\infty\in\gamma^*$. We now consider a sequence $\{p_j\}$ of points $p_j=\phi_0(t_j)$ ($t_j\in I^0$; $j=1, 2, \cdots$) of $\gamma\subset\Omega^*$. The corresponding sequence of numbers $\{t_j\}$ has a convergent subsequence. We assume for convenience that $\{t_j\}$ itself converges. If $\lim t_j\in I^0$ then $p_0=\lim p_j\in\gamma$, since ϕ_0 is a homeomorphism. On the other hand, if $\lim t_j$ is 0 or 1 then $\{p_j\}$ must, at any rate have a subsequence $\{p_{j_k}\}$ that converges to a point of the closed compact set γ^*. This limit point can only be p_∞: otherwise $\bar{\gamma}$ would have an endpoint in the region Ω, and this would contradict the separating properties of γ. Hence, every sequence $\{t_j\} \subset I^0$ with $\lim t_j = 0$ or $\lim t_j = 1$ must satisfy $\lim \phi_0(t_j) = p_\infty$. The homeomorphism ϕ defined by $\phi(t) = \phi_0(t)$ for $t \in I^0$, and $\phi(0) = \phi(1) = p_\infty$ is the required extension of ϕ_0.

THEOREM. *Let γ and δ be transversals of a simply connected region Ω whose boundary Γ is nonempty. Then the boundary relative to Ω of any component of $\Omega - (\gamma \cup \delta)$ is one of the following*: (i) *a Jordan curve,* (ii) *a transversal of Ω, or* (iii) *a pair of transversals. In cases* (i) *and* (ii) *this relative boundary is a subset of $\gamma \cup \delta$, while it actually coincides with $\gamma \cup \delta$ in case* (iii).

PROOF. We pass to the one-point compactification $\Omega^* = \Omega \cup (p_\infty)$. In view of the lemma, the transversals γ and δ are "compactified" as γ^* and δ^*, where p_∞ belongs to $\gamma^* \cap \delta^*$. By the corollary, γ^* and δ^* are Jordan curves in Ω^*.

a. Suppose that the intersection $\gamma^* \cap \delta^*$ contains at least one point besides p_∞. Since Ω^* is homeomorphic to a sphere, we conclude from Kerékjártó's theorem that every component of $\Omega^* - (\delta^* \cup \gamma^*)$ is a Jordan region. Returning to Ω we see that every component of $\Omega - (\gamma \cup \delta)$ is a simply connected region whose boundary relative to Ω either is a Jordan curve or consists of a transversal of Ω. The former possibility takes place if the boundary of the image on Ω^* does not pass through p_∞, while the latter situation holds if p_∞ does belong to the boundary of that image. Thus, we have either case (i) or case (ii).

b. On the other hand, if $\gamma^* \cap \delta^* = (p_\infty)$ then on returning to Ω there will be one residual region of $\Omega - (\gamma \cup \delta)$ that is a *quadrangle*. Its boundary will consist of γ, a connected subset of Γ, δ, and a second connected subset of Γ. Thus, case (iii) can occur. However, for a given pair γ, δ, there is at most one residual region of this type, and the other two are then of type (ii).

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