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ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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I. Of considerable significance for the stability analysis of signal transmission systems is the relation between the boundedness and asymptotic behavior of the solutions of the linear differential equation

$$\frac{dy}{dt} = A(t)y + p(t) \tag{1}$$

and of the solutions of the nonlinear equation

$$\frac{dz}{dt} = A(t)z + \phi(z;\, t). \tag{2}$$

Several results on this relationship have been obtained by Perron [1], Bellman [2], Coddington and Levinson [3] and others. The results of the present note—which we state after suitable restriction of (1) and (2)—are further theorems on this relationship.

In (1) and (2) we suppose that the $n\times n$ matrix $A(t)$ has elements which are real-valued, continuous and bounded for $t\geqq 0$ while $p(t)$, $\phi(z;\, t)$ are n-vectors with the former having elements which are real-valued and continuous for $t\geqq 0$ and the latter having elements which are real-valued and continuous for all $t\geqq 0$ and all $z\in V$, where V is some neighborhood of $z=0$ in the space of n-tuples of real numbers.

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Norms of vectors and matrices are denoted by $\|\cdot\|$ and defined as the sum of the moduli of the components. Vectors will be called convergent if their elements tend to finite limits as $t\to\infty$.

The fundamental matrix of solutions of the homogeneous equation

$$\frac{dx}{dt} = A(t)x \tag{3}$$

will be denoted by $X(t)$ where we take $X(0)=I$. We denote by $Y(t)$ the function $\int_0^t X(t)X^{-1}(\tau)d\tau$ and by $Y(\infty)$, $\lim_{t\to\infty} Y(t)$ when this limit exists. Our principal result may be stated as

THEOREM 1. *If* (i) *every solution of* (1) *is convergent for every convergent* $p(t)$;

(ii) *for sufficiently small* $\|z\|$, $\lim_{t\to\infty}\phi(z;t)=\phi(z;\infty)$;

(iii) *for sufficiently small* β, $\|\phi(0;t)\|\leqq\beta$ *for* $t\geqq 0$;

(iv) *for* $\epsilon>0$, *there exist* $\delta>0$ *and* $T\geqq 0$ *such that*

$$\|\phi(z_1;t)-\phi(z_2;t)\| \leqq \epsilon\|z_1-z_2\| \qquad \text{for } \|z_i\|\leqq\delta,\ i=1,2, \text{ and } t\geqq T;$$

then for every vector c for which $\|c\|$ *is sufficiently small a unique bounded solution* $z=z(t;T,c)$ *of* (2) *satisfying* $z(T;T,c)=c$, *exists on* $[T,\infty)$ *and all such solutions converge to the same limit vector*, ξ, *which may be determined uniquely as a solution of the equation*

$$\xi = Y(\infty)\phi(\xi;\infty).$$

II. In [4], it is shown that (i) implies all of the following:

(i′) there exist $\alpha>0$ and $K>0$ such that

$$\|X(t)X^{-1}(\tau)\| \leqq K\exp[-\alpha(t-\tau)]$$

for all $t\geqq\tau\geqq 0$;

(i″) there exists $M>0$ such that $\int_0^t\|X(t)X^{-1}(\tau)\|d\tau\leqq M$ for all $t\geqq 0$;

(i‴) $\lim_{t\to\infty} Y(t)$ exists as a matrix with finite elements.

Let us define the successive approximations, $z_n(t)$, to the solution of (2) as

$$\begin{aligned} &\frac{dz_0}{dt} = A(t)z_0+\phi(z^*;t), \qquad z_0(T)=c; \\ &\frac{dz_{n+1}}{dt} = A(t)z_{n+1}+\phi(z_n;t), \quad z_{n+1}(T)=c, \qquad n=0,1,2,\cdots, \end{aligned} \tag{4}$$

where we assume for the present only that z^* is a fixed vector. A solution $z_{n+1}(t)$ of (4) satisfies

$$z_{n+1}(t) = X(t)X^{-1}(T)c + \int_T^t X(t)X^{-1}(\tau)\phi(z_n(\tau);\tau)d\tau \tag{5}$$

from which we obtain, by (iii), (iv), (i′) and (i″), the estimate

$$\|z_{n+1}(t)\| \leqq K\|c\| + M\beta + \epsilon M \sup_{T\leqq t} \|z_n(t)\|. \tag{6}$$

If for some fixed λ, $0\leqq\lambda<1$, we suppose $\beta\leqq\lambda(1-\epsilon M)\delta/M$ and $\|c\|<(1-\lambda)(1-\epsilon M)\delta/K$ and take $\epsilon\leqq\mu M^{-1}$ for fixed μ, $0<\mu<1$, it follows by (6) that $\sup_{T\leqq t}\|z_{n+1}(t)\|<\delta$ if $\sup_{T\leqq t}\|z_n(t)\|<\delta$. An estimate similar to (6) shows that when $\|z^*\|<\delta$ then $\sup_{T\leqq t}\|z_0(t)\|<\delta$. This completes the induction and shows that our approximations are bounded on $[T, \infty)$, uniformly for $n=0, 1, 2, \cdots$.

From (5) we find that

$$\|z_{n+1}(t) - z_n(t)\| \leqq \mu \sup_{T\leqq\tau\leqq t} \|z_n(\tau) - z_{n-1}(\tau)\|;$$

hence

$$\sup_{T\leqq\tau\leqq t} \|z_{n+1}(\tau) - z_n(\tau)\| \leqq \mu \sup_{T\leqq\tau\leqq t} \|z_n(\tau) - z_{n-1}(\tau)\|$$

so that the series $\sum_{n=0}^{\infty} \sup_{T\leqq\tau\leqq t}\|z_{n+1}(\tau)-z_n(\tau)\|$ converges uniformly for $T\leqq t$. This in turn implies that the $z_{n+1}(t)$ converge to a limit vector $z(t)$ uniformly for $T\leqq t$. From (5) it then follows that $z(t)$ is the unique solution of (2) satisfying

$$z(t) = X(t)X^{-1}(T)c + \int_T^t X(t)X^{-1}(\tau)\phi(z(\tau);\tau)d\tau. \tag{7}$$

Now consider the transformation R defined on $\{z \mid \|z\|\leqq\delta\}$ by

$$Rz = Y(\infty)\phi(z;\infty); \tag{8}$$

by (iii),

$$\|Rz\| \leqq \|Y(\infty)\|(\epsilon\delta + \beta) \leqq M\epsilon\delta + \lambda(1 - \epsilon M)\delta < (1 - \lambda)\delta + \lambda\delta = \delta,$$

and by (iv),

$$\|Rz_1 - Rz_2\| \leqq \|Y(\infty)\|\epsilon\|z_1 - z_2\| \leqq \mu\|z_1 - z_2\|.$$

Since $\mu<1$, Banach's fixed point theorem implies that there exists a unique ξ, $\|\xi\|<\delta$, such that

$$\xi = R\xi.$$

Again from (5) and (iv) we find that

$$(9) \qquad \|z_{n+1}(t) - z_n(t)\| \leqq \epsilon \int_T^t \|X(t)X^{-1}(\tau)\| \, \|z_n(\tau) - z_{n-1}(\tau)\| d\tau;$$

using (i′), (i″), an argument similar to that for the sufficiency of [4, Theorem 1] shows that if $\|z_n(t) - z_{n-1}(t)\| \to 0$ as $t \to \infty$ then the integral on the right of (9) tends to zero as $t \to \infty$. Now from (i), (ii), (4) and [4, Theorem 1] it follows that $\lim_{t\to\infty} z_0(t) = Y(\infty)\phi(z^*; \infty)$; on the other hand, from (4), (5) and (iv) we have

$$\|z_1(t) - z_0(t)\| \leqq \epsilon \int_T^t \|X(t)X^{-1}(\tau)\| \, \|z_0(\tau) - z^*\| d\tau.$$

Thus, if $z^* \equiv \xi$, the integral on the right tends to zero as $t \to \infty$; this, together with the argument from (9), implies inductively that $\lim_{t\to\infty} z_n(t) = \xi$, $n = 0, 1, 2, \cdots$. Although this convergence is not necessarily uniform in n, the convergence of $z_n(t)$ to $z(t)$ is, as noted previously, uniform in t; invoking the Moore-Osgood theorem on limits we conlude finally that

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z_n(t) = \xi.$$

REMARK. The simplest way to ensure that (i) is satisfied is to take $A(t) = A + B(t)$ where the characteristic roots of A have only negative real parts and where $\|B(t)\| \to 0$ as $t \to \infty$; in this case $Y(\infty) = -A^{-1}$ [4]. One of the simplest functions satisfying (ii), (iii), (iv), is $\phi(z; t) = f(z) + g(t)$ where

(a) the bound of $\|g(t)\|$ is sufficiently small and $\lim_{t\to\infty} g(t)$ exists;
(b) $f(0) = 0$ and, for $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|f(z_1) - f(z_2)\| \leqq \epsilon \|z_1 - z_2\| \qquad \text{for } \|z_i\| \leqq \delta, \; i = 1, 2.$$

III. The next theorem shows that the existence of a fixed point of the transformation $R(z) \equiv Y(\infty)\phi(z; \infty)$ is necessary for the existence of a limit for a solution of (2).

THEOREM 2. *If conditions* (i) *and* (ii) *of Theorem* 1 *are satisfied, with the convergence in* (ii) *uniform in z, and if z(t) is a bounded solution of* (2) *for which* $\lim_{t\to\infty} z(t) = z^*$, *then* z^* *satisfies* $z^* = Y(\infty)\phi(z^*; \infty)$.

For we have

$$(10) \qquad \begin{aligned} Y(\infty)\phi(z^*; \infty) - z^* &= [z(t) - z^*] - X(t)X^{-1}(T)c \\ &\quad - \left[\int_T^t X(t)X^{-1}(\tau)d\tau - Y(\infty)\right]\phi(z^*; \infty) \\ &\quad - \int_T^t X(t)X^{-1}(\tau)[\phi(z(\tau); \tau) - \phi(z^*; \infty)]d\tau. \end{aligned}$$

By hypothesis, the first three terms on the right of (10) tend to zero as $t\to\infty$. The hypotheses, together with the Moore-Osgood theorem, imply that $\|\phi(z(t);t)-\phi(z^*;\ \infty)\|$ tends to zero as $t\to\infty$; an argument like that of [4] mentioned above then shows that the fourth term on the right of (10) tends to zero. Hence $\|Y(\infty)\phi(z^*;\ \infty)-z^*\|<\epsilon$ for every $\epsilon>0$, which proves the theorem.

In the event that, for some z_1, $\phi(z_1;\ \infty)=0$, then we may dispense with the existence of $Y(\infty)$; to be precise we have

THEOREM 3. *If* (v) *every solution of* (1) *is bounded for every bounded* $p(t)$;

(vi) *for some* z_1, *with* $\|z_1\|$ *sufficiently small,* $\lim_{t\to\infty}\phi(z_1;t)=0$;

(vii) *for* $\epsilon>0$, *there exist* $\delta>0$ *and* $T\geqq 0$ *such that*

$$\|\phi(z;t) - \phi(z_1;t)\| \leqq \epsilon\|z\| \qquad \text{for } \|z\| \leqq \delta \text{ and } t \geqq T;$$

then for every vector c for which $\|c\|$ *is sufficiently small, a bounded solution,* $z(t;\ T,c)$, *of* (2) *satisfying* $z(T;\ T,c)=c$ *exists on* $[T,\ \infty)$ *and all such solutions tend to zero as* $t\to\infty$.

As is well known (vide [4]), (v) implies (i′) so that Theorem 3 is in essence a special case of the theorem of [3, p. 327] cited previously and in turn implies the result of Bellman in [2].

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