1. Introduction. Suppose $X$ is a translation-invariant linear subspace of $C_0(R)$ (the space of all continuous functions on the real line $R$ that vanish at infinity) that is dense in $C_0(R)$ with respect to the uniform topology. If $\mu$ is a measure on the line such that

$$\int_{-\infty}^{\infty} f(x + t)d\mu(x) = \int_{-\infty}^{\infty} f(x)d\mu(x)$$

for all $f \in X$ and all $t \in R$, does it follow that $\mu$ is a constant multiple of the Lebesgue measure?

Our interest in this question arose in the following context. Let $\Gamma$ be the dual group of a locally compact abelian group $G$ (written additively), and let $(x, \gamma)$ be the value of the character $\gamma \in \Gamma$ at the point $x \in G$. If $f \in L^1(G)$, its Fourier transform is defined by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)dx \quad (\gamma \in \Gamma),$$

where $dx$ denotes the Haar measure of $G$. The inversion formula

$$f(x) = \int_{\Gamma} \hat{f}(\gamma)(x, \gamma)d\gamma \quad (x \in G),$$

where $d\gamma$ denotes the (suitably normalized) Haar measure of $\Gamma$, is valid for all $f \in P^1$, the space of all linear combinations of positive definite functions in $L^1(G)$. In two standard texts [1, p. 143; 2, p. 413], (3) is proved by first showing that there is a positive measure $\mu$ on $\Gamma$ such that

$$f(x) = \int_{\Gamma} \hat{f}(\gamma)(x, \gamma)d\mu(\gamma)$$

and

$$\int_{\Gamma} \hat{f}(\gamma)d\mu(\gamma) = \int_{\Gamma} \hat{f}(\gamma + \gamma')d\mu(\gamma)$$

for all $\gamma' \in \Gamma$ and for all $\hat{f} \in P^1$ (the set of all Fourier transforms of $L^1(G)$).
members of $P^1$). Since $P^1$ is dense in $C_0(\Gamma)$, it is concluded from (5) that $\mu$ is a Haar measure on $\Gamma$, and then (4) establishes (3).

In Theorem 1 below we show that the correctness of the italicized statement in the preceding sentence stems from the fact that $P^1$ is an algebra (under pointwise multiplication). This point is glossed over in both [1] and [2], and the reader is left with the erroneous impression that the only measures $\mu$ on $\Gamma$ that satisfy (5) for a dense subset of functions in $C_0(\Gamma)$ are the Haar measures.\footnote{We are grateful to Mr. J. A. Smoller for raising the question of how to deduce from (5) that $\mu$ is a Haar measure.} We are thus led to the following question, to which we have obtained partial answers:

Suppose $X$ is a translation-invariant subspace of $C_0(G)$, $\mu$ is a measure on $G$, and $\mu$ acts invariantly on $X$, i.e.,

\begin{equation}
\int_G f(x + t) d\mu(x) = \int_G f(x) d\mu(x) \quad (f \in X, t \in G).
\end{equation}

What information does this give about $\mu$, and what information does it give about the translation-invariant functional $T_\mu$ defined on $X$ by

\begin{equation}
T_\mu(f) = \int_G f(x) d\mu(x).
\end{equation}

By a measure we always mean a complex, countably additive, regular set function defined on the Borel sets of $G$ which is finite for all sets with compact closure. The space of all $f \in C_0(G)$ with compact support will be denoted by $C_c(G)$.

2. Uniqueness theorems.

Theorem 1. Suppose $A$ is a dense translation-invariant subalgebra of $C_0(G)$, $\mu$ is a measure on $G$, and $\int |f| d|\mu| < \infty$ for all $f \in A$. If $\mu$ acts invariantly on $A$, then $\mu$ is a constant (complex) multiple of the Haar measure of $G$.

Proof. Choose $g \in C_c(G)$. Since $A$ is dense in $C_0(G)$, $A$ contains a function $h$ which vanishes at no point of the support of $g$. Let $k = g/h$; then $k \in C_c(G)$, and so there is a sequence $\{f_n\}$ in $A$ that converges to $k$ uniformly on $G$. Since $\int |f| d|\mu| < \infty$, Lebesgue’s dominated convergence theorem shows that

\begin{equation}
\lim_{n \to \infty} \int_G f_n(x + t) h(x + t) d\mu(x) = \int_G g(x + t) d\mu(x)
\end{equation}

for every $t \in G$. Since $f_n h \in A$, the left side of (8) is independent of $t$. The same is therefore true of the right side, and we have shown that
\(\mu\) acts invariantly on \(C_c(G)\).

Since every measure on \(G\) is determined by its action on \(C_c(G)\),
the uniqueness theorem for Haar measure\(^3\) completes the proof.

**Theorem 2.** Suppose \(\mu\) is a measure on \(G\) that acts invariantly on
a translation-invariant linear subspace \(X\) of \(C_0(G)\), such that \(\int |f| \, d\mu\) < \(\infty\) and \(\int |f| \, dx < \infty\) for all \(f \in X\). If
\[
\int g(x) \, d\mu(x) = 0
\]
for some \(g \in X\), then there exists a constant \(\lambda\) such that
\[
\int f(x) \, d\mu(x) = \lambda \int f(x) \, dx \quad (f \in X).
\]

**Proof.** For any \(f \in X\), we have
\[
g(0) \int g(x) \, d\mu(x) = \int g(t) \, dt \int f(x-t) \, d\mu(x)
\]
\[
= \int d\mu(x) \int f(x-t) g(t) \, dt,
\]
by the invariant action of \(\mu\) on \(X\) and by Fubini's theorem. Since
\[
\int f(x-t) g(t) \, dt = \int g(x-t) f(t) \, dt,
\]
(11) is symmetric in \(f\) and \(g\), and so
\[
g(0) \int f(x) \, d\mu(x) = \int g(x) \, d\mu(x).
\]
This is (10), with \(\lambda = [g(0)]^{-1} \int g \, d\mu\).

**Remarks.** (a) This proof is patterned after a simple uniqueness
proof for Haar measure on abelian groups [1, p. 116].
(b) We did not assume that \(X\) is dense in \(C_0(G)\). (Cf. Theorem 3,
however.)
(c) The conclusion of the theorem amounts to the statement that
the functional \(T_\mu\) defined on \(X\) by (7) is also given by integration
with respect to a Haar measure. That is to say, \(\mu\) acts on \(X\) like a
Haar measure. This does not imply, however, that \(\mu\) is itself translation-

invariant.

For example, let \(X\) be the set of all \(f \in C_c(R)\) such that \(\int e^x f(x) e^x \, dx = 0\), and take \(d\mu(x) = (1+e^x) \, dx\). The space \(X\) is translation-invariant,
\(\mu\) acts invariantly on \(X\), and Theorem 3 below shows that \(X\) is even
dense in \(C_0(R)\).

---

\(^3\) Uniqueness of translation-invariant measures, which is customarily stated only
for positive measures, is valid for complex measures as well. For abelian groups this
follows, for example, from Theorem 2 below, with \(X = C_c(G)\).
(d) If condition (9) is omitted from Theorem 2, the conclusion is no longer valid, even if \( X \) is dense in \( C_0(G) \) and if \( \mu \) is a positive measure.

To see this, let \( X \) be the linear space generated by all translates of even functions \( f \in C_c(R) \) for which \( f(0) = 0 \), and take \( d\mu(x) = x^3dx \). Note that

\[
\int_{-\infty}^{\infty} f(x + t)x^3dx = \int_{-\infty}^{\infty} f(x)(x^3 - 2tx + t^3)dx.
\]

If \( f \) is even, then \( \int f(x)x^3dx = 0 \). Since \( \int f(x)dx = f(0) = 0 \) for all \( f \in X \), \( \mu \) acts invariantly on \( X \). But if \( f(0) = 2, f(3) = -1, f(6) = 0, f \) is linear between these points, \( f(x) = 0 \) for \( x > 6 \), and \( f \) is even, then \( f \in X \) but \( \int_{-\infty}^{\infty} x^3f(x)dx < 0 \). Thus \( \int fd\mu \) is not a constant multiple of \( f(0) \).

3. Subspaces of \( C_0(R) \). In this section, we confine our attention to the group \( R \) of real numbers. We show, first, that translation-invariant subspaces of \( C_0(R) \) are usually dense.

**Theorem 3.** If a subspace \( X \) of \( C_0(R) \) contains all translates of some nonzero function \( f \) with compact support, then \( X \) is dense in \( C_0(R) \).

**Proof.** Suppose \( \mu \) is a bounded measure on \( R \) that annihilates \( X \); then

\[
\int_{-\infty}^{\infty} f(x - t)d\mu(x) = 0 \quad (t \in R).
\]

If \( F \) is the Fourier transform of \( \hat{f} \), where \( \hat{f}(x) = f(-x) \), (13) implies that \( F \cdot \hat{\mu} = 0 \). But \( F \) is an analytic function, hence has only isolated zeros, and since \( \hat{\mu} \) is continuous, we conclude that \( \hat{\mu} = 0 \). By the uniqueness theorem for Fourier-Stieltjes transforms, \( \mu = 0 \), and so \( X \) is dense in \( C_0(R) \), by the Hahn-Banach theorem.

**Remark.** We did not really need to assume that \( f \in C_c(R) \). All we needed was that the zeros of \( \hat{f} \) should form a nowhere dense set.

The examples following Theorem 2 show that a measure that acts invariantly on a translation-invariant subspace \( X \) of \( C_c(R) \) need not be a constant multiple of Lebesgue measure. We have already remarked, however, that a translation-invariant functional (7) on \( X \) is necessarily a multiple of the Lebesgue integral in case some member of \( X \) has a nonzero integral. In the next theorem, we recapture the uniqueness property of translation-invariant functionals on \( X \) even when every member of \( X \) has zero integral. The proof is similar to that of Theorem 2.

**Theorem 4.** Suppose \( \mu \) is a measure on \( R \) that acts invariantly on a translation-invariant linear subspace \( X \) of \( C_c(R) \). Let \( p \) be the smallest nonnegative integer such that
(14) \[ \int_{-\infty}^{\infty} x^p g(x) dx \neq 0 \]

for some \( g \in X \). Then there is a constant \( \lambda \) such that

(15) \[ \int_{-\infty}^{\infty} f(x) d\mu(x) = \lambda \int_{-\infty}^{\infty} x^p f(x) dx \quad (f \in X). \]

Note that the right side of (15) is a constant times the \( p \)th derivative of \( f \) at the origin.

**Proof.** The case \( p = 0 \) is dealt with in Theorem 2.

Suppose \( p > 0 \), \( f \in X \), \( f \neq 0 \), and the support of \( f \) is contained in the interval \([-A, A]\). Let \( f_0 = f \), and for \( k \geq 1 \), let \( f_k(x) = \int_{-A}^{x} f_{k-1}(t) dt \). By induction, one obtains the well-known formula

(16) \[ f_k(x) = \frac{1}{(k-1)!} \int_{-A}^{x} (x-t)^{k-1} f(t) dt \quad (k = 1, 2, \cdots). \]

We have chosen \( p \) so that

(17) \[ \int_{-A}^{A} f_{k-1}(t) dt = f_k(A) = \frac{1}{(k-1)!} \int_{-A}^{A} (A-t)^{k-1} f(t) dt = 0 \]

for \( 1 \leq k \leq p \). Thus, \( f_0, f_1, \cdots, f_p \) all have compact support. Furthermore,

(18) \[ f_0(0) = \int_{-\infty}^{\infty} f_p(t) dt = f_{p+1}(A) = \frac{1}{p!} \int_{-\infty}^{\infty} (A-t)^p f(t) dt \]

\[ = \frac{(-1)^p}{p!} \int_{-\infty}^{\infty} p f(t) dt. \]

Hence, if \( g \) is a function in \( X \) that satisfies (14), then \( g_p(0) \neq 0 \).

Now, as in the proof of Theorem 2, for any \( f \in X \),

(19) \[ g_p(0) \int_{-\infty}^{\infty} f(x) d\mu(x) = \int_{-\infty}^{\infty} d\mu(x) \int_{-\infty}^{\infty} g_p(t) f(x-t) dt. \]

Integration by parts \( p \) times yields

(20) \[ \int_{-\infty}^{\infty} g_p(t) f(x-t) dt = \int_{-\infty}^{\infty} g(t) f_p(x-t) dt = \int_{-\infty}^{\infty} f_p(t) g(x-t) dt, \]

and comparison of (19) and (20) shows that

(21) \[ g_p(0) \int_{-\infty}^{\infty} f(x) d\mu(x) = f_p(0) \int_{-\infty}^{\infty} g(x) d\mu(x). \]
With \( g \) fixed so that \( g_p(0) \neq 0 \), (21) together with (18) yields (15) and completes the proof.

By virtue of Theorem 4 every measure that acts invariantly on a translation-invariant subspace \( X \) of \( C_c(R) \) differs from a measure \( \lambda x^\rho dx \), for suitable \( \lambda \) and \( \rho \), by a measure that vanishes on all of \( X \). The theorem also furnishes some nonzero measures that vanish on \( X \), namely, the measures \( x^k dx \) for \( 0 \leq k < \rho \). Moreover, it provides a clue for finding still other such measures.

For every \( f \in X \), the Fourier transform \( \hat{f} \) of \( f \) can be extended to an entire function in the complex plane. Associate with each complex number \( \alpha \) a nonnegative integer \( m(\alpha) \), the largest integer \( k \) such that \( \hat{f}(z)(z - \alpha)^{-k} \) is regular at \( z = \alpha \) for all \( f \in X \), and let \( E \) be the set of all \( \alpha \) for which \( m(\alpha) > 0 \). If \( X \neq \{ 0 \} \), it is clear that \( E \) has no limit point in the finite plane. An equivalent definition of \( m(\alpha) \) is that

\[
\int_{-\infty}^{\infty} x^k e^{-iax} f(x) dx = 0 \quad \text{for } 0 \leq k < m(\alpha) \text{ and all } f \in X; \quad \text{whereas}
\]

\[
\int_{-\infty}^{\infty} x^{m(\alpha)} e^{-iax} g(x) dx \neq 0 \quad \text{for some } g \in X. \]

If now \( \alpha \in E \) and

\[
d\mu(x) = (c_0 + c_1 x + \cdots + c_k x^k) e^{-iax} dx \quad (k < m(\alpha)),
\]

then (22) implies that

\[
\int_{-\infty}^{\infty} f(x) d\mu(x) = 0 \quad \text{for all } f \in X.
\]

To sum up, any finite linear combination of measures (24) and the measure \( x^{m(0)} dx \) acts invariantly on \( X \), and so do certain infinite sums. For instance, if \( \alpha_1, \alpha_2, \alpha_3, \cdots \) are points of \( E \) and if \( \{ c_j \} \) tends to 0 rapidly enough, the series

\[
s(x) = \sum_{j=1}^{\infty} c_j e^{-i\alpha_j x}
\]

converges uniformly on compact subsets of \( R \), and the measure \( d\mu(x) = s(x) dx \) acts invariantly on \( X \).

Both examples which follow Theorem 2 are of this sort. It is noteworthy that the complex zeros of the Fourier transforms play a role here.
We conclude with an example of a space $X \subset C_c(R)$ (a dense subspace of $C_c(R)$, by Theorem 3) which contains a nontrivial nonnegative function and on which a positive measure (not a Haar measure) acts invariantly: $X$ consists of all $f\in C_c(R)$ with $f(1)=f(-1)=0$, and $d\mu(x) = (2+\sin x)dx$. Since $2i \sin x = e^{ix} - e^{-ix}$, $\int_{-\infty}^{\infty} f(x) \sin x \, dx = 0$ for all $f \in X$, and so $\mu$ acts invariantly on $X$; also, $X$ contains the nonnegative triangular function $f$ defined by

$$f(x) = \max (2\pi - |x|, 0) \quad (-\infty < x < \infty).$$

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BARRELLED SPACES AND THE OPEN MAPPING THEOREM

TAQDIR HUSAIN AND MARK MAHOWALD

1. Introduction. If $E$ and $F$ are any two topological vector spaces then the following statement may or may not be true:

   (A) If $f$ is any linear and continuous mapping of $E$ onto $F$ then $f$ is open.

   It is well known [1] that (A) is true when $E$ and $F$ are Fréchet spaces. An extension due to Pták [6], and Robertson and Robertson [7] is that (A) is true if $E$ is $B$-complete and $F$ is barrelled ($t$-space). We ask here whether these results characterize Fréchet and $B$-complete spaces respectively. More precisely, let $\mathcal{F}$ and $\mathcal{S}$ denote the classes of all Fréchet and barrelled spaces respectively. We ask if a topological vector space $E$, having the property that (A) is true whenever $F \in \mathcal{F}(\mathcal{S})$, is necessarily a Fréchet ($B$-complete) space.

   A well-known example of an LF-space and a theorem of Dieudonné and Schwartz [5, Theorem 1] supplies a counterexample to the above for $\mathcal{F}$. Here, we give an example showing that the other case is also false.

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