1. Introduction. Suppose $X$ is a translation-invariant linear subspace of $C_0(R)$ (the space of all continuous functions on the real line $R$ that vanish at infinity) that is dense in $C_0(R)$ with respect to the uniform topology. If $\mu$ is a measure on the line such that

$$\int_{-\infty}^{\infty} f(x + t)d\mu(x) = \int_{-\infty}^{\infty} f(x)d\mu(x)$$

for all $f \in X$ and all $t \in R$, does it follow that $\mu$ is a constant multiple of the Lebesgue measure?

Our interest in this question arose in the following context. Let $\Gamma$ be the dual group of a locally compact abelian group $G$ (written additively), and let $(x, \gamma)$ be the value of the character $\gamma \in \Gamma$ at the point $x \in G$. If $f \in L^1(G)$, its Fourier transform is defined by

$$\hat{f}(\gamma) = \int_{G} f(x)(-x, \gamma)dx \quad (\gamma \in \Gamma),$$

where $dx$ denotes the Haar measure of $G$. The inversion formula

$$f(x) = \int_{\Gamma} \hat{f}(\gamma)(x, \gamma)d\gamma \quad (x \in G),$$

where $d\gamma$ denotes the (suitably normalized) Haar measure of $\Gamma$, is valid for all $f \in P^1$, the space of all linear combinations of positive definite functions in $L^1(G)$. In two standard texts [1, p. 143; 2, p. 413], (3) is proved by first showing that there is a positive measure $\mu$ on $\Gamma$ such that

$$f(x) = \int_{\Gamma} \hat{f}(\gamma)(x, \gamma)d\mu(\gamma)$$

and

$$\int_{\Gamma} \hat{f}(\gamma)d\mu(\gamma) = \int_{\Gamma} \hat{f}(\gamma + \gamma')d\mu(\gamma)$$

for all $\gamma' \in \Gamma$ and for all $\hat{f} \in \hat{P}^1$ (the set of all Fourier transforms of $P^1$).
members of $P^1$). Since $P^1$ is dense in $C_0(\Gamma)$, it is concluded from (5) that $\mu$ is a Haar measure on $\Gamma$, and then (4) establishes (3).

In Theorem 1 below we show that the correctness of the italicized statement in the preceding sentence stems from the fact that $P^1$ is an algebra (under pointwise multiplication). This point is glossed over in both [1] and [2], and the reader is left with the erroneous impression that the only measures $\mu$ on $\Gamma$ that satisfy (5) for a dense subset of functions in $C_0(\Gamma)$ are the Haar measures. We are thus led to the following question, to which we have obtained partial answers:

Suppose $X$ is a translation-invariant subspace of $C_0(G)$, $\mu$ is a measure on $G$, and $\mu$ acts invariantly on $X$, i.e.,

$$(6) \quad \int_G f(x+t)d\mu(x) = \int_G f(x)d\mu(x) \quad (f \in X, t \in G).$$

What information does this give about $\mu$, and what information does it give about the translation-invariant functional $T_\mu$ defined on $X$ by

$$(7) \quad T_\mu(f) = \int_G f(x)d\mu(x)?$$

By a measure we always mean a complex, countably additive, regular set function defined on the Borel sets of $G$ which is finite for all sets with compact closure. The space of all $f \in C_0(G)$ with compact support will be denoted by $C_c(G)$.

2. Uniqueness theorems.

**Theorem 1.** Suppose $A$ is a dense translation-invariant subalgebra of $C_0(G)$, $\mu$ is a measure on $G$, and $\int |f|d|\mu| < \infty$ for all $f \in A$. If $\mu$ acts invariantly on $A$, then $\mu$ is a constant (complex) multiple of the Haar measure of $G$.

**Proof.** Choose $g \in C_c(G)$. Since $A$ is dense in $C_0(G)$, $A$ contains a function $h$ which vanishes at no point of the support of $g$. Let $k = g/h$; then $k \in C_c(G)$, and so there is a sequence $\{f_n\}$ in $A$ that converges to $k$ uniformly on $G$. Since $\int |h|d|\mu| < \infty$, Lebesgue's dominated convergence theorem shows that

$$(8) \quad \lim_{n \to \infty} \int_G f_n(x+t)h(x+t)d\mu(x) = \int_G g(x+t)d\mu(x)$$

for every $t \in G$. Since $f_nh \in A$, the left side of (8) is independent of $t$. The same is therefore true of the right side, and we have shown that

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We are grateful to Mr. J. A. Smoller for raising the question of how to deduce from (5) that $\mu$ is a Haar measure.
μ acts invariantly on \( C_c(G) \).

Since every measure on \( G \) is determined by its action on \( C_c(G) \), the uniqueness theorem for Haar measure\(^4\) completes the proof.

**Theorem 2.** Suppose \( μ \) is a measure on \( G \) that acts invariantly on a translation-invariant linear subspace \( X \) of \( C_0(G) \), such that \( \int |f| \, dμ < \infty \) and \( \int |f| \, dx < \infty \) for all \( f \in X \). If

\[
\hat{g}(0) \neq 0
\]

for some \( g \in X \), then there exists a constant \( λ \) such that

\[
\int_{g} f(x) \, dμ(x) = λ \int_{g} f(x) \, dx \quad (f \in X).
\]

**Proof.** For any \( f \in X \), we have

\[
\begin{align*}
\hat{g}(0) \int_{g} f \, dμ &= \int_{g} g(t) \, dt \int_{g} f(x - t) \, dμ(x) \\
&= \int_{g} dμ(x) \int_{g} f(x - t) g(t) \, dt,
\end{align*}
\]

by the invariant action of \( μ \) on \( X \) and by Fubini’s theorem. Since \( \int f(x - t) g(t) \, dt = \int g(x - t) f(t) \, dt \), (11) is symmetric in \( f \) and \( g \), and so

\[
\hat{g}(0) \int_{g} f \, dμ = \hat{f}(0) \int_{g} g \, dμ.
\]

This is (10), with \( λ = [\hat{g}(0)]^{-1} \int_{g} g \, dμ \).

**Remarks.** (a) This proof is patterned after a simple uniqueness proof for Haar measure on abelian groups \([1, \text{p. 116}]\).

(b) We did not assume that \( X \) is dense in \( C_0(G) \). (Cf. Theorem 3, however.)

(c) The conclusion of the theorem amounts to the statement that the functional \( T_μ \) defined on \( X \) by (7) is also given by integration with respect to a Haar measure. That is to say, \( μ \) acts on \( X \) like a Haar measure. This does not imply, however, that \( μ \) is itself translation-invariant.

For example, let \( X \) be the set of all \( f \in C_c(R) \) such that \( \int_{-\infty}^{\infty} f(x) e^{tx} \, dx = 0 \), and take \( dμ(x) = (1 + e^x) \, dx \). The space \( X \) is translation-invariant, \( μ \) acts invariantly on \( X \), and Theorem 3 below shows that \( X \) is even dense in \( C_0(R) \).

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\(^4\) Uniqueness of translation-invariant measures, which is customarily stated only for positive measures, is valid for complex measures as well. For abelian groups this follows, for example, from Theorem 2 below, with \( X = C_c(G) \).
(d) If condition (9) is omitted from Theorem 2, the conclusion is no longer valid, even if $X$ is dense in $C_0(G)$ and if $\mu$ is a positive measure.

To see this, let $X$ be the linear space generated by all translates of even functions $f \in C_c(\mathbb{R})$ for which $f(0) = 0$, and take $d\mu(x) = x^2 dx$. Note that

$$\int_{-\infty}^{\infty} f(x + t) x^2 dx = \int_{-\infty}^{\infty} f(x)(x^2 - 2tx + t^2) dx.$$ 

If $f$ is even, then $\int f(x) x^2 dx = 0$. Since $\int f(x) dx = \hat{f}(0) = 0$ for all $f \in X$, $\mu$ acts invariantly on $X$. But if $\hat{f}(0) = 2$, $\hat{f}(3) = -1$, $\hat{f}(6) = 0$, $f$ is linear between these points, $f(x) = 0$ for $x > 6$, and $f$ is even, then $f \in X$ but $\int_{-\infty}^{\infty} x^2 f(x) dx < 0$. Thus $\int f d\mu$ is not a constant multiple of $\hat{f}(0)$.

3. Subspaces of $C_0(\mathbb{R})$. In this section, we confine our attention to the group $R$ of real numbers. We show, first, that translation-invariant subspaces of $C_0(\mathbb{R})$ are usually dense.

**Theorem 3.** If a subspace $X$ of $C_0(R)$ contains all translates of some nonzero function $f$ with compact support, then $X$ is dense in $C_0(\mathbb{R})$.

**Proof.** Suppose $\mu$ is a bounded measure on $R$ that annihilates $X$; then

$$\int_{-\infty}^{\infty} f(x - t) d\mu(x) = 0 \quad (t \in \mathbb{R}).$$

If $F$ is the Fourier transform of $\hat{f}$, where $\hat{f}(x) = \hat{f}(-x)$, (13) implies that $F \cdot \mu = 0$. But $F$ is an analytic function, hence has only isolated zeros, and since $\mu$ is continuous, we conclude that $\hat{\mu} = 0$. By the uniqueness theorem for Fourier-Stieltjes transforms, $\mu = 0$, and so $X$ is dense in $C_0(\mathbb{R})$, by the Hahn-Banach theorem.

**Remark.** We did not really need to assume that $f \in C_c(\mathbb{R})$. All we needed was that the zeros of $\hat{f}$ should form a nowhere dense set.

The examples following Theorem 2 show that a measure that acts invariantly on a translation-invariant subspace $X$ of $C_c(\mathbb{R})$ need not be a constant multiple of Lebesgue measure. We have already remarked, however, that a translation-invariant functional (7) on $X$ is necessarily a multiple of the Lebesgue integral in case some member of $X$ has a nonzero integral. In the next theorem, we recapture the uniqueness property of translation-invariant functionals on $X$ even when every member of $X$ has zero integral. The proof is similar to that of Theorem 2.

**Theorem 4.** Suppose $\mu$ is a measure on $R$ that acts invariantly on a translation-invariant linear subspace $X$ of $C_c(\mathbb{R})$. Let $p$ be the smallest nonnegative integer such that
for some \( g \in X \). Then there is a constant \( \lambda \) such that
\[
\int_{-\infty}^{\infty} f(x) d\mu(x) = \lambda \int_{-\infty}^{\infty} x^p f(x) dx \quad (f \in X).
\]

Note that the right side of (15) is a constant times the \( p \)th derivative of \( f \) at the origin.

**Proof.** The case \( p = 0 \) is dealt with in Theorem 2.

Suppose \( p > 0 \), \( f \in X \), \( f \neq 0 \), and the support of \( f \) is contained in the interval \([-A, A]\). Let \( f_0 = f \), and for \( k \geq 1 \), let \( f_k(x) = \int_A^x f_{k-1}(t) dt \). By induction, one obtains the well-known formula
\[
f_k(x) = \frac{1}{(k - 1)!} \int_{-A}^{x} (x - t)^{k-1} f(t) dt \quad (k = 1, 2, \cdots).
\]

We have chosen \( p \) so that
\[
\int_{-A}^{A} f_{k-1}(t) dt = f_k(A) = \frac{1}{(k - 1)!} \int_{-A}^{A} (A - t)^{k-1} f(t) dt = 0
\]
for \( 1 \leq k \leq p \). Thus, \( f_0, f_1, \cdots, f_p \) all have compact support. Furthermore,
\[
f_p(0) = \int_{-\infty}^{\infty} f_p(t) dt = f_{p+1}(A) = \frac{1}{p!} \int_{-\infty}^{\infty} (A - t)^{p} f(t) dt
\]
\[
= \frac{(-1)^p}{p!} \int_{-\infty}^{\infty} t^p f(t) dt.
\]

Hence, if \( g \) is a function in \( X \) that satisfies (14), then \( \hat{g}_p(0) \neq 0 \).

Now, as in the proof of Theorem 2, for any \( f \in X \),
\[
\hat{g}_p(0) \int_{-\infty}^{\infty} f(x) d\mu(x) = \int_{-\infty}^{\infty} d\mu(x) \int_{-\infty}^{\infty} g_p(t) f(x - t) dt.
\]

Integration by parts \( p \) times yields
\[
\int_{-\infty}^{\infty} g_p(t) f(x - t) dt = \int_{-\infty}^{\infty} g(t) f_p(x - t) dt = \int_{-\infty}^{\infty} f_p(t) g(x - t) dt,
\]
and comparison of (19) and (20) shows that
\[
\hat{g}_p(0) \int_{-\infty}^{\infty} f(x) d\mu(x) = f_p(0) \int_{-\infty}^{\infty} g(x) d\mu(x).
\]
With $g$ fixed so that $g_p(0) \neq 0$, (21) together with (18) yields (15) and completes the proof.

By virtue of Theorem 4 every measure that acts invariantly on a translation-invariant subspace $X$ of $C_c(R)$ differs from a measure $\lambda x^p dx$, for suitable $\lambda$ and $p$, by a measure that vanishes on all of $X$. The theorem also furnishes some nonzero measures that vanish on $X$, namely, the measures $x^k dx$ for $0 \leq k < p$. Moreover, it provides a clue for finding still other such measures.

For every $f \in X$, the Fourier transform $\hat{f}$ of $f$ can be extended to an entire function in the complex plane. Associate with each complex number $\alpha$ a nonnegative integer $m(\alpha)$, the largest integer $k$ such that $\hat{f}(\alpha)(\alpha-\alpha)^{-k}$ is regular at $\alpha$ for all $f \in X$, and let $E$ be the set of all $\alpha$ for which $m(\alpha) > 0$. If $X \neq \{0\}$, it is clear that $E$ has no limit point in the finite plane. An equivalent definition of $m(\alpha)$ is that

$$\int_{-\infty}^{\infty} x^k e^{-iax} f(x) dx = 0$$
for $0 \leq k < m(\alpha)$ and all $f \in X$; whereas

$$\int_{-\infty}^{\infty} x^{m(\alpha)} e^{-iax} g(x) dx \neq 0$$
for some $g \in X$. If now $\alpha \in E$ and

$$d\mu(x) = (c_0 + c_1 x + \cdots + c_k x^k) e^{-iax} dx$$ \hspace{1cm} (k < m(\alpha)),

then (22) implies that

$$\int_{-\infty}^{\infty} f(x) d\mu(x) = 0$$
for all $f \in X$.

To sum up, any finite linear combination of measures (24) and the measure $x^{m(0)} dx$ acts invariantly on $X$, and so do certain infinite sums. For instance, if $\alpha_1, \alpha_2, \alpha_3, \cdots$ are points of $E$ and if $\{c_j\}$ tends to 0 rapidly enough, the series

$$s(x) = \sum_{j=1}^{\infty} c_j e^{-ia_j x}$$
converges uniformly on compact subsets of $R$, and the measure $d\mu(x) = s(x) dx$ acts invariantly on $X$.

Both examples which follow Theorem 2 are of this sort. It is noteworthy that the complex zeros of the Fourier transforms play a role here.
We conclude with an example of a space \( X \subseteq C_c(R) \) (a dense subspace of \( C_0(R) \), by Theorem 3) which contains a nontrivial nonnegative function and on which a positive measure (not a Haar measure) acts invariantly: \( X \) consists of all \( f \in C_c(R) \) with \( \mathcal{F}(1) = \mathcal{F}(-1) = 0 \), and \( d\mu(x) = (2 + \sin x)dx \). Since \( 2i \sin x = e^{ix} - e^{-ix} \), \( \int_{-\infty}^{\infty} f(x) \sin x dx = 0 \) for all \( f \in X \), and so \( \mu \) acts invariantly on \( X \); also, \( X \) contains the nonnegative triangular function \( f \) defined by

\[
(27) \quad f(x) = \max \left( 2\pi - |x|, 0 \right) \quad (-\infty < x < \infty).
\]

**References**


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**BARRELLED SPACES AND THE OPEN MAPPING THEOREM**

Taqdir Husain and Mark Mahowald

1. **Introduction.** If \( E \) and \( F \) are any two topological vector spaces then the following statement may or may not be true:

   (A) If \( f \) is any linear and continuous mapping of \( E \) onto \( F \) then \( f \) is open.

   It is well known [1] that (A) is true when \( E \) and \( F \) are Fréchet spaces. An extension due to Pták [6], and Robertson and Robertson [7] is that (A) is true if \( E \) is \( B \)-complete and \( F \) is barrelled (\( B \)-space). We ask here whether these results characterize Fréchet and \( B \)-complete spaces respectively. More precisely, let \( \mathcal{F} \) and \( \mathcal{B} \) denote the classes of all Fréchet and barrelled spaces respectively. We ask if a topological vector space \( E \), having the property that (A) is true whenever \( F \in \mathcal{F}(\mathcal{B}) \), is necessarily a Fréchet (\( B \)-complete) space.

   A well-known example of an LF-space and a theorem of Dieudonné and Schwartz [5, Theorem 1] supplies a counterexample to the above for \( \mathcal{F} \). Here, we give an example showing that the other case is also false.

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