TRANSLATION-INVARIANT FUNCTIONALS[1]

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1. Introduction. Suppose X is a translation-invariant linear subspace of $C_0(R)$ (the space of all continuous functions on the real line R that vanish at infinity) that is dense in $C_0(R)$ with respect to the uniform topology. If μ is a measure on the line such that

$$\int_{-\infty}^{\infty} f(x+t)d\mu(x) = \int_{-\infty}^{\infty} f(x)d\mu(x) \tag{1}$$

for all $f \in X$ and all $t \in R$, does it follow that μ is a constant multiple of the Lebesgue measure?

Our interest in this question arose in the following context. Let Γ be the dual group of a locally compact abelian group G (written additively), and let (x, γ) be the value of the character $\gamma \in \Gamma$ at the point $x \in G$. If $f \in L^1(G)$, its Fourier transform is defined by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)dx \qquad (\gamma \in \Gamma), \tag{2}$$

where dx denotes the Haar measure of G. The inversion formula

$$f(x) = \int_\Gamma \hat{f}(\gamma)(x, \gamma)d\gamma \qquad (x \in G), \tag{3}$$

where $d\gamma$ denotes the (suitably normalized) Haar measure of Γ, is valid for all $f \in P^1$, the space of all linear combinations of positive definite functions in $L^1(G)$. In two standard texts [1, p. 143; 2, p. 413], (3) is proved by first showing that there is a positive measure μ on Γ such that

$$f(x) = \int_\Gamma \hat{f}(\gamma)(x, \gamma)d\mu(\gamma) \tag{4}$$

and

$$\int_\Gamma \hat{f}(\gamma)d\mu(\gamma) = \int_\Gamma \hat{f}(\gamma + \gamma')d\mu(\gamma) \tag{5}$$

for all $\gamma' \in \Gamma$ and for all $\hat{f} \in \hat{P}^1$ (the set of all Fourier transforms of

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members of P^1). *Since* $\hat{P}^1$ *is dense in* $C_0(\Gamma)$, *it is concluded from* (5) *that* μ *is a Haar measure on* Γ, and then (4) establishes (3).

In Theorem 1 below we show that the correctness of the italicized statement in the preceding sentence stems from the fact that $\hat{P}^1$ is an *algebra* (under pointwise multiplication). This point is glossed over in both [1] and [2], and the reader is left with the erroneous impression that the only measures μ on Γ that satisfy (5) for a dense subset of functions in $C_0(\Gamma)$ are the Haar measures.[2] We are thus led to the following question, to which we have obtained partial answers:

Suppose X *is a translation-invariant subspace of* $C_0(G)$, μ *is a measure on* G, *and* μ *acts invariantly on* X, *i.e.*,

$$\int_G f(x+t)d\mu(x) = \int_G f(x)d\mu(x) \qquad (f \in X,\, t \in G). \tag{6}$$

What information does this give about μ, *and what information does it give about the translation-invariant functional* T_μ *defined on* X *by*

$$T_\mu(f) = \int_G f(x)d\mu(x)? \tag{7}$$

By a *measure* we always mean a complex, countably additive, regular set function defined on the Borel sets of G which is finite for all sets with compact closure. The space of all $f \in C_0(G)$ with compact support will be denoted by $C_c(G)$.

2. Uniqueness theorems.

THEOREM 1. *Suppose* A *is a dense translation-invariant subalgebra of* $C_0(G)$, μ *is a measure on* G, *and* $\int |f| d|\mu| < \infty$ *for all* $f \in A$. *If* μ *acts invariantly on* A, *then* μ *is a constant* (*complex*) *multiple of the Haar measure of* G.

PROOF. Choose $g \in C_c(G)$. Since A is dense in $C_0(G)$, A contains a function h which vanishes at no point of the support of g. Let $k = g/h$; then $k \in C_c(G)$, and so there is a sequence $\{f_n\}$ in A that converges to k uniformly on G. Since $\int |h| d|\mu| < \infty$, Lebesgue's dominated convergence theorem shows that

$$\lim_{n\to\infty} \int_G f_n(x+t)h(x+t)d\mu(x) = \int_G g(x+t)d\mu(x) \tag{8}$$

for every $t \in G$. Since $f_n h \in A$, the left side of (8) is independent of t. The same is therefore true of the right side, and we have shown that

[2] We are grateful to Mr. J. A. Smoller for raising the question of how to deduce from (5) that μ is a Haar measure.

μ acts invariantly on $C_c(G)$.

Since every measure on G is determined by its action on $C_c(G)$, the uniqueness theorem for Haar measure[3] completes the proof.

THEOREM 2. *Suppose μ is a measure on G that acts invariantly on a translation-invariant linear subspace X of $C_0(G)$, such that $\int |f| d|\mu| < \infty$ and $\int |f| dx < \infty$ for all $f \in X$. If*

$$\hat{g}(0) \neq 0 \tag{9}$$

for some $g \in X$, then there exists a constant λ such that

$$\int_G f(x) d\mu(x) = \lambda \int_G f(x) dx \qquad (f \in X). \tag{10}$$

PROOF. For any $f \in X$, we have

$$\begin{aligned} \hat{g}(0) \int_G f d\mu &= \int_G g(t) dt \int_G f(x - t) d\mu(x) \\ &= \int_G d\mu(x) \int_G f(x - t) g(t) dt, \end{aligned} \tag{11}$$

by the invariant action of μ on X and by Fubini's theorem. Since $\int f(x-t)g(t)dt = \int g(x-t)f(t)dt$, (11) is symmetric in f and g, and so

$$\hat{g}(0) \int_G f d\mu = \hat{f}(0) \int_G g d\mu. \tag{12}$$

This is (10), with $\lambda = [\hat{g}(0)]^{-1} \int_G g d\mu$.

REMARKS. (a) This proof is patterned after a simple uniqueness proof for Haar measure on abelian groups [1, p. 116].

(b) We did not assume that X is dense in $C_0(G)$. (Cf. Theorem 3, however.)

(c) The conclusion of the theorem amounts to the statement that the functional T_μ defined on X by (7) is also given by integration with respect to a Haar measure. That is to say, μ acts *on* X like a Haar measure. *This does not imply, however, that μ is itself translation-invariant.*

For example, let X be the set of all $f \in C_c(R)$ such that $\int_{-\infty}^{\infty} f(x) e^x dx = 0$, and take $d\mu(x) = (1 + e^x) dx$. The space X is translation-invariant, μ acts invariantly on X, and Theorem 3 below shows that X is even dense in $C_0(R)$.

[3] Uniqueness of translation-invariant measures, which is customarily stated only for positive measures, is valid for complex measures as well. For abelian groups this follows, for example, from Theorem 2 below, with $X = C_c(G)$.

(d) *If condition* (9) *is omitted from Theorem* 2, *the conclusion is no longer valid, even if* X *is dense in* $C_0(G)$ *and if* μ *is a positive measure.*

To see this, let X be the linear space generated by all translates of even functions $f \in C_c(R)$ for which $\hat{f}(0)=0$, and take $d\mu(x)=x^2dx$. Note that

$$\int_{-\infty}^{\infty} f(x+t)x^2dx = \int_{-\infty}^{\infty} f(x)(x^2-2tx+t^2)dx.$$

If f is even, then $\int f(x)xdx=0$. Since $\int f(x)dx=\hat{f}(0)=0$ for all $f \in X$, μ acts invariantly on X. But if $f(0)=2, f(3)=-1, f(6)=0, f$ is linear between these points, $f(x)=0$ for $x>6$, and f is even, then $f \in X$ but $\int_{-\infty}^{\infty} x^2 f(x)dx<0$. Thus $\int f d\mu$ is not a constant multiple of $\hat{f}(0)$.

3. **Subspaces of** $C_0(R)$. In this section, we confine our attention to the group R of real numbers. We show, first, that translation-invariant subspaces of $C_0(R)$ are usually dense.

THEOREM 3. *If a subspace* X *of* $C_0(R)$ *contains all translates of some nonzero function* f *with compact support, then* X *is dense in* $C_0(R)$.

PROOF. Suppose μ is a bounded measure on R that annihilates X; then

$$\int_{-\infty}^{\infty} f(x-t)d\mu(x) = 0 \qquad (t \in R). \tag{13}$$

If F is the Fourier transform of $\bar{f}$, where $\bar{f}(x)=f(-x)$, (13) implies that $F\cdot\hat{\mu}=0$. But F is an analytic function, hence has only isolated zeros, and since $\hat{\mu}$ is continuous, we conclude that $\hat{\mu}=0$. By the uniqueness theorem for Fourier-Stieltjes transforms, $\mu=0$, and so X is dense in $C_0(R)$, by the Hahn-Banach theorem.

REMARK. We did not really need to assume that $f \in C_c(R)$. All we needed was that the zeros of $\hat{f}$ should form a nowhere dense set.

The examples following Theorem 2 show that a measure that acts invariantly on a translation-invariant subspace X of $C_c(R)$ need not be a constant multiple of Lebesgue measure. We have already remarked, however, that a translation-invariant functional (7) on X is necessarily a multiple of the Lebesgue integral in case some member of X has a nonzero integral. In the next theorem, we recapture the uniqueness property of translation-invariant functionals on X even when every member of X has zero integral. The proof is similar to that of Theorem 2.

THEOREM 4. *Suppose* μ *is a measure on* R *that acts invariantly on a translation-invariant linear subspace* X *of* $C_c(R)$. *Let* p *be the smallest nonnegative integer such that*

$$\int_{-\infty}^{\infty} x^p g(x)dx \neq 0 \tag{14}$$

for some $g \in X$. *Then there is a constant* λ *such that*

$$\int_{-\infty}^{\infty} f(x)d\mu(x) = \lambda \int_{-\infty}^{\infty} x^p f(x)dx \qquad (f \in X). \tag{15}$$

Note that the right side of (15) is a constant times the pth derivative of $\hat{f}$ at the origin.

PROOF. The case $p=0$ is dealt with in Theorem 2.

Suppose $p>0, f\in X, f\neq 0$, and the support of f is contained in the interval $[-A, A]$. Let $f_0=f$, and for $k\geqq 1$, let $f_k(x)=\int_{-A}^{x} f_{k-1}(t)dt$. By induction, one obtains the well-known formula

$$f_k(x) = \frac{1}{(k-1)!}\int_{-A}^{x} (x-t)^{k-1}f(t)dt \qquad (k = 1, 2, \cdots). \tag{16}$$

We have chosen p so that

$$\int_{-A}^{A} f_{k-1}(t)dt = f_k(A) = \frac{1}{(k-1)!}\int_{-A}^{A} (A-t)^{k-1}f(t)dt = 0 \tag{17}$$

for $1\leqq k\leqq p$. Thus, $f_0, f_1, \cdots, f_p$ all have compact support. Furthermore,

$$\begin{aligned} \hat{f}_p(0) &= \int_{-\infty}^{\infty} f_p(t)dt = f_{p+1}(A) = \frac{1}{p!}\int_{-\infty}^{\infty} (A-t)^p f(t)dt \\ &= \frac{(-1)^p}{p!}\int_{-\infty}^{\infty} t^p f(t)dt. \end{aligned} \tag{18}$$

Hence, if g is a function in X that satisfies (14), then $\hat{g}_p(0)\neq 0$.

Now, as in the proof of Theorem 2, for any $f\in X$,

$$\hat{g}_p(0)\int_{-\infty}^{\infty} f(x)d\mu(x) = \int_{-\infty}^{\infty} d\mu(x)\int_{-\infty}^{\infty} g_p(t)f(x-t)dt. \tag{19}$$

Integration by parts p times yields

$$\int_{-\infty}^{\infty} g_p(t)f(x-t)dt = \int_{-\infty}^{\infty} g(t)f_p(x-t)dt = \int_{-\infty}^{\infty} f_p(t)g(x-t)dt, \tag{20}$$

and comparision of (19) and (20) shows that

$$\hat{g}_p(0)\int_{-\infty}^{\infty} f(x)d\mu(x) = \hat{f}_p(0)\int_{-\infty}^{\infty} g(x)d\mu(x). \tag{21}$$

With g fixed so that $\hat{g}_p(0) \neq 0$, (21) together with (18) yields (15) and completes the proof.

By virtue of Theorem 4 every measure that acts invariantly on a translation-invariant subspace X of $C_c(R)$ differs from a measure $\lambda x^p dx$, for suitable λ and p, by a measure that vanishes on all of X. The theorem also furnishes some nonzero measures that vanish on X, namely, the measures $x^k dx$ for $0 \leqq k < p$. Moreover, it provides a clue for finding still other such measures.

For every $f \in X$, the Fourier transform $\hat{f}$ of f can be extended to an entire function in the complex plane. Associate with each complex number α a nonnegative integer $m(\alpha)$, the largest integer k such that $\hat{f}(z)(z-\alpha)^{-k}$ is regular at $z=\alpha$ for all $f \in X$, and let E be the set of all α for which $m(\alpha) > 0$. If $X \neq \{0\}$, it is clear that E has no limit point in the finite plane. An equivalent definition of $m(\alpha)$ is that

$$\int_{-\infty}^{\infty} x^k e^{-i\alpha x} f(x) dx = 0 \tag{22}$$

for $0 \leqq k < m(\alpha)$ and all $f \in X$; whereas

$$\int_{-\infty}^{\infty} x^{m(\alpha)} e^{-i\alpha x} g(x) dx \neq 0 \tag{23}$$

for some $g \in X$. If now $\alpha \in E$ and

$$d\mu(x) = (c_0 + c_1 x + \cdots + c_k x^k) e^{-i\alpha x} dx \qquad (k < m(\alpha)), \tag{24}$$

then (22) implies that

$$\int_{-\infty}^{\infty} f(x) d\mu(x) = 0 \tag{25}$$

for all $f \in X$.

To sum up, any finite linear combination of measures (24) and the measure $x^{m(0)} dx$ acts invariantly on X, and so do certain infinite sums. For instance, if $\alpha_1, \alpha_2, \alpha_3, \cdots$ are points of E and if $\{c_j\}$ tends to 0 rapidly enough, the series

$$s(x) = \sum_{j=1}^{\infty} c_j e^{-i\alpha_j x} \tag{26}$$

converges uniformly on compact subsets of R, and the measure $d\mu(x) = s(x)dx$ acts invariantly on X.

Both examples which follow Theorem 2 are of this sort. It is noteworthy that the *complex* zeros of the Fourier transforms play a role here.

We conclude with an example of a space $X \subset C_c(R)$ (a dense subspace of $C_0(R)$, by Theorem 3) which contains a nontrivial nonnegative function and on which a positive measure (not a Haar measure) acts invariantly: X consists of all $f \in C_c(R)$ with $\hat{f}(1) = \hat{f}(-1) = 0$, and $d\mu(x) = (2 + \sin x)dx$. Since $2i \sin x = e^{ix} - e^{-ix}$, $\int_{-\infty}^{\infty} f(x) \sin x dx = 0$ for all $f \in X$, and so μ acts invariantly on X; also, X contains the nonnegative triangular function f defined by

$$f(x) = \max(2\pi - |x|, 0) \qquad (-\infty < x < \infty). \tag{27}$$

References

1. L. H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953.

2. M. A. Naimark, *Normed rings*, Noordhoff, Groningen, 1959.

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BARRELLED SPACES AND THE OPEN MAPPING THEOREM[1]

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1. **Introduction.** If E and F are any two topological vector spaces then the following statement may or may not be true:

(A) If f is any linear and continuous mapping of E onto F then f is open.

It is well known [1] that (A) is true when E and F are Fréchet spaces. An extension due to Pták [6], and Robertson and Robertson [7] is that (A) is true if E is B-complete and F is barrelled (t-space). We ask here whether these results characterize Fréchet and B-complete spaces respectively. More precisely, let $\mathfrak{F}$ and $\mathfrak{J}$ denote the classes of all Fréchet and barrelled spaces respectively. We ask if a topological vector space E, having the property that (A) is true whenever $F \in \mathfrak{F}(\mathfrak{J})$, is necessarily a Fréchet (B-complete) space.

A well-known example of an LF-space and a theorem of Dieudonné and Schwartz [5, Theorem 1] supplies a counterexample to the above for $\mathfrak{F}$. Here, we give an example showing that the other case is also false.

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