A NOTE ON TRANSLATION INVARIANTS

ROY L. ADLER AND ALAN G. KONHEIM

1. **Introduction.** Autocorrelation functions play a central role in many engineering applications. The following operation describes a certain optical system [2]: from a function f of two variables, representing the transparency of a photographic slide, another transparency function F is produced on a photographic plate which is the autocorrleation function of f; that is,

$$F(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)f(\xi - x, \eta - y)\, d\xi\, d\eta.$$

The essential feature of the function F is that it remains unchanged if f is replaced by f^* where $f^*(\xi, \eta) = f(\xi+\alpha, \eta+\beta)$. A uniqueness question now arises; if f and g have the same autocorrelation function, are they related by a rectilinear motion of the plane?

The answer to this question is negative as is shown in §4 by counterexample. In §2 we present this problem in a more general setting and in §3 the main theorem.

2. **Notation and definitions.** Let G be a locally compact abelian group, μ Haar measure, and $L_{1,r}(G)$ the space of real-valued μ-integrable functions on G. For each $s\in G$ define the translation operator $T_s\colon L_{1,r}(G)\to L_{1,r}(G)$ by $(T_sf)(x) = f(sx)$ and denote by $\mathfrak{T}(f)$ the set $\{g\colon g\in L_{1,r}(G),\ (T_sf)(x) = g(x)(\mu \text{ a.e.}) \text{ for some } s\in G\}$ of translates of f. A functional $\rho(\cdot)$ with domain $L_{1,r}(G)$ will be called *translation invariant* if $\rho(f) = \rho(g)$ for all $g\in\mathfrak{T}(f)$. A class of translation invariants $\mathfrak{P} = \{\rho_\omega(\cdot)\colon \omega\in\Omega\}$ is *complete* if $\rho_\omega(f) = \rho_\omega(g)$ for all $\omega\in\Omega$ implies $g\in\mathfrak{T}(f)$.

The *kth order autocorrelation function* of $f\in L_{1,r}(G)$ is the real-valued function $\rho_k(f) \equiv \rho_k(f)(x_1, x_2, \cdots, x_k)$ with domain $G^{(k)} = G\times G\times \cdots \times G$ (k copies) defined formally by

$$\rho_k(f)(x_1, x_2, \cdots, x_k) = \int_G f(\xi)f(\xi x_1)f(\xi x_2) \cdots f(\xi x_k)\mu(d\xi).$$

The proof that $\rho_k(f)$ is $\mu^{(k)}$-almost everywhere[1] defined and in $L_{1,r}(G)$ is a modification of the argument of [1, p. 121] used in establishing that the convolution of integrable functions is integrable. The map-

Presented to the Society, January 23, 1961; received by the editors December 10, 1960 and, in revised form, March 20, 1961.

[1] $\mu^{(k)}$ denotes the direct product measure on $G^{(k)}$ given by μ on G.

ping $f \to \rho_k(f)$ of $L_{1,r}(G)$ into $L_{1,r}(G^{(k)})$ is a translation invariant functional as μ is Haar measure.

Let $\hat{G}$ denote the character group of G. For $h \in L_{1,r}(G^{(k)})$ define the "k-dimensional" Fourier transform by

$$\hat{h}(\chi_1, \cdots, \chi_k) = \int_{G^{(k)}} \prod_{i=1}^{k} \overline{\chi_i}(\xi_i) h(\xi_1, \cdots, \xi_k) \mu^{(k)}(d\xi_1 \times \cdots \times d\xi_k)$$

where $\chi_i \in \hat{G}$ and $^-$ denotes complex conjugation. For $f \in L_{1,r}(G)$ let $\sigma(f) = \{\chi : \chi \in \hat{G}, \hat{f}(\chi) \neq 0\}$.

If K is a group and H a subset of K, we shall say that a mapping ϕ, $\phi: H \to C = \{\exp(2\pi i x): 0 \leqq x < 1\}$ *acts homomorphically on H to C* if

(i) $\phi(a^{-1}) = [\phi(a)]^{-1}$ whenever $a, a^{-1} \in H$.

(ii) For each N, $\phi(\prod_{i=1}^{N} a_i) = \prod_{i=1}^{N} \phi(a_i)$ whenever $a_1, a_2, \cdots, a_N$ and $\prod_{i=1}^{N} a_i$ are in H.

3. Main theorem.

THEOREM. *$\{\rho_k(\cdot): k = 1, 2, \cdots\}$ is a complete set of translation invariants for $L_{1,r}(G)$.*

LEMMA 1. *Let H be a symmetric subset of a group K (i.e., if $a \in H$ then $a^{-1} \in H$). If ϕ acts homomorphically on H to C then ϕ can be extended to a homomorphism of the subgroup $[H]$, generated by H, to C.*

PROOF. For $c \in [H]$ define $\phi(c) = \phi(a_1)\phi(a_2) \cdots \phi(a_p)$ where $c = a_1 a_2 \cdots a_p$, $a_i \in H (1 \leqq i \leqq p)$. To see that this definition of $\phi(c)$ is consistent let us suppose $c = b_1 b_2 \cdots b_q$ with $b_i \in H (1 \leqq i \leqq q)$. Then $b_q = a_1 a_2 \cdots a_p b_1^{-1} b_2^{-1} \cdots b_{q-1}^{-1}$ so that $\phi(b_q) = \phi(a_1)\phi(a_2) \cdots \phi(a_p)\phi(b_1^{-1})\phi(b_2^{-1}) \cdots \phi(b_{q-1}^{-1})$ by (ii) above. By (i) $\phi(b_i^{-1}) = [\phi(b_i)]^{-1}$ so that $\phi(b_1)\phi(b_2) \cdots \phi(b_q) = \phi(a_1)\phi(a_2) \cdots \phi(a_p)$. The extended mapping is easily seen to be a homomorphism of $[H]$ into C.

LEMMA 2. *If K is a topological group, H an open symmetric subset of K and ϕ a continuous mapping $\phi: H \to C$ which acts homomorphically on H to C, then ϕ can be extended to a continuous homomorphism of the (necessarily closed) subgroup $[H]$, generated by H, to C.*

PROOF. By Lemma 1, ϕ can be extended to ϕ^* an algebraic homomorphism of $[H]$ to C. It suffices to verify that ϕ^* is continuous. Let H_n denote the set $H \cdot H \cdot \cdots \cdot H$ (n copies);[2] H_n is clearly an open subset of K. Let ϕ_n denote the restriction of ϕ^* to H_n. Clearly ϕ_n is continuous on H_n and since $[H] = \bigcup_{n=1}^{\infty} H_n$, ϕ^* is continuous on $[H]$. As $[H]$ is an open subgroup of K it follows that it is necessarily a closed subgroup [3, p. 37].

[2] $H_n = \{c: c = a_1 a_2 \cdots a_n, a_i \in H\}$.

PROOF OF THE THEOREM. If $f \in L_{1,r}(G)$ then

$$\hat{f}(\chi^{-1}) = \overline{\hat{f}(\chi)}$$

and hence $\sigma(f)$ is a symmetric subset of $\hat{G}$. An elementary computation using Fubini's theorem yields

$$\hat{\rho}_k(f)(\chi_1, \chi_2, \cdots, \chi_k) = \hat{f}\left(\prod_{i=1}^{k} \chi_i^{-1}\right) \prod_{i=1}^{k} \hat{f}(\chi_i) \tag{1}$$

Suppose f, $g \in L_{1,r}(G)$ with

$$\rho_k(f) = \rho_k(g), \qquad k = 1, 2, \cdots. \tag{2}$$

From (2) with $k = 1$ we obtain

$$\hat{f}(\chi)\hat{f}(\chi^{-1}) = |\hat{f}(\chi)|^2 = |\hat{g}(\chi)|^2 = \hat{g}(\chi)\hat{g}(\chi^{-1}) \tag{3}$$

so that we may write

$$\hat{g}(\chi) = \phi(\chi)f(\chi) \tag{4}$$

where $|\phi(\chi)| = 1$ for $\chi \in \sigma(f)$. The reality of f and g and (4) implies moreover

$$\phi(\chi^{-1}) = \overline{\phi(\chi)} = [\phi(\chi)]^{-1} \tag{5}$$

again for $\chi \in \sigma(f)$. Equations (1), (2), (4) and (5) yield

$$\phi\left(\prod_{i=1}^{N} \chi_i\right) = \prod_{i=1}^{N} \phi(\chi_i) \tag{6}$$

whenever

$$\chi_1, \chi_2, \cdots, \chi_N, \prod_{i=1}^{N} \chi_i \in \sigma(f).$$

Since $\hat{f}(\cdot)$ is continuous and does not vanish on $\sigma(f)$, $\phi(\cdot)$ is continuous on $\sigma(f)$ and $\sigma(f)$ is open. Thus ϕ is continuous and acts homomorphically on the open symmetric subset $\sigma(f)$ to C. By Lemma 2, ϕ can be extended to a continuous homomorphism of $[\sigma(f)]$ to C. The mapping ϕ: $[\sigma(f)] \to C$ can be extended [**4**, p. 138] to a character of $\hat{G}$. Thus we have

$$\hat{g}(\chi) = \phi(\chi)\hat{f}(\chi) \tag{7}$$

for all $\chi \in \hat{G}$, with ϕ a character of $\hat{G}$. By the duality theorem [**1**, p. 151]

$$\phi(\chi) = \chi(a) \tag{8}$$

for some fixed $a \in G$ and therefore the uniqueness theorem [**1**, p. 146] finally yields

$$g(x) = f(ax) \quad (\mu \text{ a.e.}). \tag{9}$$

4. Counterexamples. We will now exhibit a counterexample to the conjecture

$$\rho_1(f) = \rho_1(g) \Rightarrow g \in \mathfrak{T}(f).$$

Let G be the real line and A a subset of G of finite positive measure which is not a translate of its inverse set. If f and g are the characteristic functions of A and $-A=\{-x: x\in A\}$ respectively then $\rho_1(f)(x)=\mu(A\cap(A-x))=\mu(-A\cap(-A-x))=\rho_1(g)(x)$ while $g\notin\mathfrak{T}(f)$. Similarly the conjecture

$$\rho_1(f) = \rho_1(g) \Rightarrow g \in \mathfrak{T}(f) \cap \mathfrak{T}(f^-)$$

where $f^-(x)=f(x^{-1})$ is also false. To show this let A and B be subsets of the real line as described above and f and g the characteristic functions of $A\times B$ and $A\times(-B)$. Then f and g have the same (first order) autocorrelation function while $g\notin\mathfrak{T}(f)\cup\mathfrak{T}(f^-)$.

It is likewise conjectured by the authors that no finite subset of $\{\rho_k(\cdot): k=1, 2, \cdots\}$ is a complete set of translation invariants unless restrictions are placed upon the functions. In this connection the following corollary is easily obtained.

COROLLARY. *If* $\mathfrak{U}_\nu$ *is the subset of* $L_{1,r}(G)$ *consisting of those* f *for which* $\sigma(f)\cdot\sigma(f)\cdot \cdots \cdot\sigma(f)$ (ν *copies*) *is a subgroup of* $\hat{G}$ *then* $\{\rho_k(\cdot): k=1, 2, \cdots, 3\nu-1\}$ *is a complete set of translation invariants for* $\mathfrak{U}_\nu$. *In particular if* $\hat{f}(\chi)\neq 0$ $(\chi\in\hat{G})$ *and* $\rho_i(f)=\rho_i(g)$ $(i=1, 2)$ *then* $g\in\mathfrak{T}(f)$.

5. Further considerations. For complex-valued μ-integrable functions $\sigma(f)$ is no longer symmetric in general and our argument fails. In fact if G is the unit interval and $f\in L_1(G)$ with

$$\int_0^1 f(\xi)\exp(-2\pi i n\xi)d\xi = 0 \begin{cases} n < 0, \\ n > N \end{cases}$$

then $\rho_k(f)\equiv 0$ for all k. Minor modifications of the definition of the autocorrelation function do not substantially change the situation.

REFERENCES

1. L. H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953.

2. W. Meyer-Eppler and G. Darius, *Two-dimensional photographic autocorrelation of pictures and alphabet letters*, Proc. 3rd London Sympos. Information Theory, pp. 34–36.

3. D. Montgomery and L. Zippen, *Topological transformation groups*, Interscience, New York, 1955.

4. L. Pontrjagin, *Topological groups*, Princeton Univ. Press, Princeton, N. J., 1946.

IBM RESEARCH CENTER, YORKTOWN HEIGHTS, NEW YORK