ON THE FUNCTIONAL EQUATION $f(t)=g(t)-g(2t)$

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The purpose of this note is to prove the following

THEOREM. *Let f denote a periodic function of period* 1, *satisfying in* $\langle 0, 1\rangle$ *the Hölder condition of order* $\alpha>0$, *and let* $\int_0^1 f(t)dt=0$. *Then the condition*

$$\lim_{n\to\infty}\frac{1}{n}\int_0^1\left[\sum_{i=0}^{n} f(2^i t)\right]^2 dt = 0 \tag{1}$$

is necessary and sufficient for the existence of a periodic (*of period* 1) *function* $g\in L^2_{\langle 0,1\rangle}$ *such that*

$$f(t) = g(t) - g(2t). \tag{2}$$

This theorem, without the assertion that $g\in L^2_{\langle 0,1\rangle}$ was apparently established by R. Fortet in his paper [1] though his proof is not clear to me. A proof of this theorem in the case $\alpha>1/2$, was given by M. Kac [2]. Moreover, it is shown in [2] that there exists a continuous periodic (of period 1) function f satisfying (1) for which there is no function $g\in L_{\langle 0,1\rangle}$ such that (2) holds.

The method of the proof used here is related to that given by M. Kac. The real difference is in using Haar functions instead of trigonometrical functions. The Lemmas 3 and 4 were already proved for $\alpha>1/2$ in [2].

Throughout this note we will consider only measurable and periodic (of period 1) functions. Now, we define the Haar functions as follows:

$$\chi_1(t) \equiv 1,\ \chi_{2^n+k}(t) = \begin{cases} (2^n)^{1/2} \text{ in } \left(\dfrac{2k-2}{2^{n+1}}, \dfrac{2k-1}{2^{n+1}}\right), \\ -(2^n)^{1/2} \text{ in } \left(\dfrac{2k-1}{2^{n+1}}, \dfrac{2k}{2^{n+1}}\right), \\ 0 \quad \text{otherwise in } \langle 0, 1\rangle; \end{cases}$$

where $n=0, 1, \cdots; k=1, \cdots, 2^n$.

LEMMA 1. *If f satisfies the Hölder condition of order $\alpha>0$ in $\langle 0, 1\rangle$, then*

$$a_n = O\left(\frac{1}{n^{\alpha+1/2}}\right), \tag{3}$$

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where $a_n=\int_0^1 f(t)\chi_n(t)dt$.

The proof follows from the obvious formula

$$a_{2^n+k}=\frac{1}{2(2^{n/2})}\int_0^1\left[f\left(\frac{t}{2^{n+1}}+\frac{2k-2}{2^{n+1}}\right)-f\left(\frac{t}{2^{n+1}}+\frac{2k-1}{2^{n+1}}\right)\right]dt,$$

$n=0, 1, \cdots ; k=1, \cdots, 2^n$, and the fact that f satisfies Hölder's condition.

LEMMA 2. *Let f be integrable in $\langle 0, 1\rangle$ and let*

$$f(t)=\sum_{n=1}^{\infty}a_n\chi_n(t).$$

Then

$$f(2t)=\sum_{n=1}^{\infty}b_n\chi_n(t),$$

where

$$b_1=a_1,\ b_2=0,\ b_{2^n+k}=\begin{cases}\dfrac{1}{2^{1/2}}a_{2^{n-1}+k} & \text{for } 1\leqq k\leqq 2^{n-1},\\[2ex] \dfrac{1}{2^{1/2}}a_k & \text{for } 2^{n-1}<k\leqq 2^n,\end{cases}\tag{4}$$

for $n=1, 2, \cdots$.

REMARK. We write equality between f and its Fourier-Haar series because this series converges almost everywhere to f (see [3]).

The proof is immediate because of the formulas

$$\chi_1(2t)=\chi_1(t),\quad \chi_{2^n+k}(2t)=\frac{1}{2^{1/2}}\chi_{2^{n+1}+k}(t)+\frac{1}{2^{1/2}}\chi_{2^{n+1}+2^n+k}(t),$$

where $n=0, 1, \cdots ; k=1, \cdots, 2^n$.

LEMMA 3. *If f satisfies the Hölder condition of order $\alpha>0$ in $\langle 0, 1\rangle$ and $\int_0 f(t)dt=0$, then*

$$\int_0^1 f(t)f(2^it)dt=O\left(\frac{1}{2^{i\alpha}}\right)\tag{5}$$

where O depends on f and α.

PROOF. Using (4) it is easy to establish by induction the following formulae for Fourier-Haar coefficients $\{c_n\}$ of the function $f(2^it)$, where $i\geqq 0$:

$$
(6)\qquad \begin{aligned} c_\nu &= 0 \qquad \text{for } \nu \leqq 2^i, \\ c_{2^n+k} &= \frac{1}{2^{i/2}}\, a_{2^{n-i}+q^{(i)}{}_{2^n+k}}, \end{aligned}
$$

where $n \geqq i$; $k = 1, \cdots, 2^n$; and $q^{(i)}_{2^n+k}$ are some integers such that $1 \leqq q^{(i)}_{2^n+k} \leqq 2^{n-i}$. Now, by the Parseval's identity, we have

$$
\int_0^1 f(t)f(2^it)dt = \sum_{n=i}^{\infty}\sum_{k=1}^{2^n} \frac{1}{2^{i/2}}\, a_{2^n+k}a_{2^{n-i}+q^{(i)}_{2^n+k}};
$$

hence by (3) we get

$$
(7)\qquad \frac{1}{2^{i/2}}\sum_{k=1}^{2^n} \left| a_{2^n+k}a_{2^{n-i}+q^{(i)}_{2^n+k}} \right| = O\left(\frac{2^{i\alpha}}{2^{2n\alpha}}\right).
$$

Finally, from (7)

$$
\int_0^1 f(t)f(2^it)dt = \sum_{n=i}^{\infty} O\left(\frac{2^{i\alpha}}{2^{2n\alpha}}\right) = O\left(\frac{1}{2^{i\alpha}}\right).
$$

LEMMA 4. *If the function f satisfies the conditions of Lemma* 3 *and the condition* (1), *then the identity*

$$
(8)\qquad \sum_{i=0}^{\infty}\int_0^1 f(t)f(2^it)dt = \frac{1}{2}\int_0^1 f^2(t)dt
$$

holds.

The easy proof will be omitted.

PROOF OF THE THEOREM. The necessity is trivial and we are going to prove sufficiency.

Let us define the numbers $\{c_\nu\}$, $\nu = 1, 2, \cdots$, by the following formulae:

$$
(9)\qquad c_1 = 0,\ a_2 = c_2,\ a_{2^n+k} = \begin{cases} c_{2^n+k} - \dfrac{1}{2^{1/2}}\, c_{2^{n-1}+k} & \text{for } 1 \leqq k \leqq 2^{n-1}, \\[2ex] c_{2^n+k} - \dfrac{1}{2^{1/2}}\, c_k & \text{for } 2^{n-1} < k \leqq 2^n, \end{cases}
$$

where $n = 1, 2, \cdots$; and $\{a_\nu\}$ are the Fourier-Haar coefficients of the function f given in our theorem. We see from Lemma 2 that it is enough to show that

$$
(10)\qquad \sum_{\nu=2}^{\infty} c_\nu^2 < \infty,
$$

because the function g will be given by $g(t)=\sum_{n=2}^{\infty} c_\nu\chi_\nu(t)$. From (9) we get immediately

$$c_{2^n+k} = \sum_{i=0}^{n} \frac{1}{2^{i/2}} a_{2^{n-i}+q_{2^n+k}^{(i)}} \tag{11}$$

for $n=0, 1, \cdots; k=1, \cdots, 2^n$; where $q_{2^n+k}^{(i)}$ are the same as in (6). Now, let us put

$$S_{2^{n+1}}[t; f(2^i\tau)] = \sum_{\nu=2}^{2^{n+1}} \int_0^1 f(2^i\tau)\chi_\nu(\tau)d\tau\chi_\nu(t).$$

Using Lemma 2 we get

$$S_{2^{n+1}}[t; f(2^i\tau)] = S_{2^n}[2t; f(2^{i-1}\tau)] \tag{12}$$

for $i\geqq 1$. By (6) we have, for $i\geqq 0$,

$$f(2^it) = \frac{1}{2^{i/2}} \sum_{n=i}^{\infty}\sum_{k=1}^{2^n} a_{2^{n-i}+q_{2^n+k}^{(i)}}\chi_{2^n+k}(t). \tag{13}$$

Now, by (11), (12) and (13) we have

$$\begin{aligned}
\sum_{\nu=2}^{2^{n+1}} c_\nu^2 &= \int_0^1 \left\{ \sum_{i=0}^{n} S_{2^{n+1}}[\tau; f(2^it)] \right\}^2 d\tau \\
&= \sum_{i=0}^{n} \int_0^1 S_{2^{n+1}}^2[\tau; f(2^it)]d\tau \\
&\quad + 2\sum_{i<j} \int_0^1 S_{2^{n+1}}[\tau; f(2^it)] S_{2^{n+1}}[\tau; f(2^jt)]d\tau \\
&= \sum_{i=0}^{n} \int_0^1 S_{2^{n+1-i}}^2[2^i\tau; f(t)]d\tau \\
&\quad + 2\sum_{i<j} \int_0^1 S_{2^{n+1-i}}[2^i\tau; f(t)] S_{2^{n+1-i}}[2^i\tau; f(2^{j-i}t)]d\tau \\
&= (n+1)\int_0^1 S_{2^{n+1}}^2[\tau; f(t)]d\tau \\
&\quad + 2\sum_{i<j} \int_0^1 S_{2^{n+1}}[\tau; f(t)] S_{2^{n+1}}[\tau; f(2^{j-i}t)]d\tau \\
&\quad - \sum_{i=1}^{n} \sum_{\nu=n+1-i}^{n} \sum_{k=1}^{2^\nu} a_{2^\nu+k}^2 \\
&\quad - 2\sum_{0<i<j} \sum_{\nu=n+1-i}^{n} \sum_{k=1}^{2^\nu} a_{2^\nu+k} \frac{1}{2^{(j-i)/2}} a_{2^{\nu-(j-i)}+q_{2^\nu+k}^{(j-i)}}.
\end{aligned}$$

First of all we shall estimate the last two sums. According to (7) we have

$$\sum_{i=1}^{n}\sum_{\nu=n+1-i}^{n}\sum_{k=1}^{2^\nu} a_{2^n+\nu}^2 = \sum_{i=1}^{n}\sum_{\nu=n+1-i}^{n} O\left(\frac{1}{2^{2\alpha\nu}}\right)$$

$$= \sum_{i=1}^{n} O\left(\frac{1}{2^{2\alpha n-i}}\right) = O(1)$$

and

$$\sum_{0<i<j}\sum_{\nu=n+1-i}^{n}\sum_{k=1}^{2^\nu} a_{2^\nu+k}\frac{1}{2^{(j-i)/2}}\, a_{2^{\nu-(j-i)}+q_{2^n+k}^{(j-i)}}$$

$$= \sum_{0<i<j}\sum_{\nu=n+1-i}^{n} O\left(\frac{2^{(j-i)\alpha}}{2^{2\alpha\nu}}\right)$$

$$= \sum_{0<i<j} O\left(\frac{2^{(j+i)\alpha}}{2^{2\alpha n}}\right)$$

$$= O(1).$$

Thus, we get

$$\sum_{\nu=2}^{2^{n+1}} c_\nu^2 = O(1) + (n+1)\int_0^1 S_{2^{n+1}}^2[\tau; f(t)]\,d\tau$$

$$+ 2\sum_{i<j}\int_0^1 S_{2^{n+1}}[\tau; f(t)]S_{2^{n+1}}[\tau; f(2^{j-i}t)]\,d\tau$$

$$= O(1) + 2\sum_{k=0}^{n}\left\{\sum_{i=0}^{k}\int_0^1 S_{2^{n+1}}[\tau; f(t)]S_{2^{n+1}}[\tau; f(2^i t)]\,d\tau - \frac{1}{2}\int_0^1 S_{2^{n+1}}^2[\tau; f(t)]\,d\tau\right\}$$

$$= O(1) + 2\sum_{k=0}^{n}\left\{\sum_{i=0}^{k}\frac{1}{2^{i/2}}\sum_{\nu=i}^{n}\sum_{\mu=1}^{2^\nu} a_{2^\nu+\mu}a_{2^{\nu-i}+q_{2^\nu+\mu}^{(i)}} - \frac{1}{2}\sum_{\nu=2}^{2^{n+1}} a_\nu^2\right\}.$$

Now we shall apply the identity (8). According to Parseval's identity it may be rewritten in the following form

$$\sum_{i=0}^{\infty}\frac{1}{2^{i/2}}\sum_{\nu=i}^{\infty}\sum_{\mu=1}^{2^\nu} a_{2^\nu+\mu}a_{2^{\nu-i}+q_{2^\nu+\mu}^{(i)}} - \frac{1}{2}\sum_{\nu=2}^{\infty} a_\nu^2 = 0.$$

Moreover, if we use (7) we get

$$\sum_{i=0}^{k} \frac{1}{2^{i/2}} \sum_{\nu=i}^{n} \sum_{\mu=1}^{2^\nu} a_{2^\nu+\mu} a_{2^{\nu-i}+q^{(i)}_{2^\nu+\mu}} - \frac{1}{2} \sum_{\nu=2}^{2^{n+1}} a_\nu^2$$
$$= - \sum_{i=0}^{k} \frac{1}{2^{i/2}} \sum_{\nu=n+1}^{\infty} \sum_{\mu=1}^{2^\nu} a_{2^\nu+\mu} a_{2^{\nu-i}+q^{(i)}_{2^\nu+\mu}}$$
$$- \sum_{i=k+1}^{\infty} \frac{1}{2^{i/2}} \sum_{\nu=i}^{\infty} \sum_{\mu=1}^{2^\nu} a_{2^\nu+\mu} a_{2^{\nu-i}+q^{(i)}_{2^\nu+\mu}} + \frac{1}{2} \sum_{\nu=2^{n+1}+1}^{\infty} a_\nu^2$$
$$= \sum_{i=0}^{k} \sum_{\nu=n+1}^{\infty} O\left(\frac{2^{\alpha i}}{2^{2\alpha\nu}}\right) + \sum_{i=k+1}^{\infty} \sum_{\nu=i}^{\infty} O\left(\frac{2^{i\alpha}}{2^{2\alpha\nu}}\right) + O\left(\frac{1}{2^{2\alpha n}}\right)$$
$$= \sum_{i=0}^{k} O\left(\frac{2^{\alpha i}}{2^{2\alpha n}}\right) + \sum_{i=k+1}^{\infty} O\left(\frac{1}{2^{\alpha i}}\right) + O\left(\frac{1}{2^{2\alpha n}}\right)$$
$$= O\left(\frac{2^{\alpha k}}{2^{2\alpha n}}\right) + O\left(\frac{1}{2^{k\alpha}}\right) + O\left(\frac{1}{2^{2\alpha n}}\right).$$

So we get at last

$$\sum_{\nu=2}^{2^{n+1}} c_\nu^2 = O(1) + \sum_{k=0}^{n} O\left(\frac{2^{k\alpha}+1}{2^{2\alpha n}} + \frac{1}{2^{k\alpha}}\right) = O(1).$$

References

1. R. Fortet, *Sur une suite egalement répartie*, Studia Math. **9** (1940), 54–70.

2. M. Kac, *On the distribution of values of sums of type* $\Sigma f(2^k t)$, Ann. of Math. **47** (1946), 33–49.

3. S. Kaczmarz und H. Steinhaus, *Theorie der Orthogonalreihen*, Warsaw, 1935.

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