AXIOMS THAT DEFINE SEMI-METRIC, MOORE, AND METRIC SPACES

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In [1] L. F. McAuley asked the following question: *is it possible to partition . . . Moore's metrization theorem into three or more parts which begins with a condition for a topological space and which ends with a condition for a metrizable space, but with necessary and sufficient conditions somewhere between these extremes for semi-metric and Moore spaces?* Axiom Z, stated below, is such a partitioning. The notation "Axiom Z_i" denotes parts (1), (2), $\cdots$, (i) of Axiom Z. In §1 it is proved in Theorems 1, 2, and 3, respectively, that a necessary and sufficient condition for a topological space to be semi-metrizable, a Moore space, and metrizable is that it satisfy Axiom Z_2, Axiom Z_3, and Axiom Z_4 respectively. A counter-example is given in §2 which shows that the argument for the statement *a Moore space is a semi-metric topological space* in Theorem 6.2 in [1] is not correct. Finally, in §3 it is shown that part (3) of Theorem 2 in [2] can be changed so that the resulting statement is equivalent to a Moore space. Definitions are given in [1].

DEFINITION. If $\{J_n\}$ denotes a sequence such that for each natural number n, J_n denotes a collection of neighborhoods covering a point set M, then the sequence $\{B_i\}$, where i denotes a natural number, is said to be a *basic refinement of* $\{J_n\}$ *for* M provided that with each point p in M there is associated a sequence $\{b_i(p)\}$ such that for each i: (1) $b_i(p)$ is a neighborhood in $\{J_n\}$, (2) $b_{i+1}(p)$ is a subset of $b_i(p)$, (3) p is the only point common to $\{b_i(p)\}$, and (4) B_i denotes the collection of all neighborhoods $b_i(p)$ for all points in M.

1. Axiom Z.

AXIOM Z. *Let T denote a topological space in which there exists a sequence $\{J_n\}$ such that:*

(1) *for each natural number n, J_n denotes a collection of neighborhoods in T covering T,*

(2) *there exists a basic refinement $\{B_n\}$ of $\{J_n\}$ for T such that if M denotes a point set and p denotes a point, then either* (a) *every neighborhood containing p contains a point in $(M-p)$ or* (b) *there exists an n such that if x denotes a point in $(M-p)$, $b_n(p)$ does not contain x and $b_n(x)$ does not contain p,*

(3) *if R denotes a neighborhood containing p and x is in R, then there*

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exists an n such that if g denotes a neighborhood in J_n that contains p, then $\bar{g}$ is a subset of R not containing x unless x is p, and

(4) *if R contains p and x is different from p, then there exists an n such that if each of h and k denotes a neighborhood in J_n, k contains p, and h and k have a common part, then h is a subset of R not containing x.*

Parts (1) and (4) are Moore's metrization theorem and parts (1) and (3) are parts (1) and (3) of Moore's Axiom 1_3.

THEOREM 1. *A necessary and sufficient condition for a topological space T to be semi-metrizable is that T satisfy Axiom Z_2.*

PROOF OF NECESSITY. It follows from the definition of a semi-metric topological space that for each point p in T, there exists a sequence $\{b_n(p)\}$ such that for each n, (1) $b_n(p)$ denotes a neighborhood containing p, (2) $b_n(p)$ is a subset of the $1/n$-neighborhood of p, and (3) $b_{n+1}(p)$ is a subset of $b_n(p)$. Now, for each n, let B_n denote the collection of all the neighborhoods $b_n(p)$ for the various points in T. Finally, for natural numbers i and m let J_m denote the collection of all the elements in B_i for all $i \geqq m$. It is not difficult to verify that Axiom Z_2 is satisfied by the sequences $\{J_n\}$ and $\{B_n\}$.

PROOF OF SUFFICIENCY. For each pair of points p and x in T, let n denote the smallest i such that $b_i(x)$ does not contain p and $b_i(p)$ does not contain x where $b_i(x)$ and $b_i(p)$ belong to B_i. Now let $d(p, x) = d(x, p) = 1/n$. Define $d(p, p) = 0$. It follows that d is a semi-metric for T and that the sufficiency of the condition is established.

THEOREM 2. *A necessary and sufficient condition for a topological space T to be a Moore space is that T satisfy Axiom Z_3.*

PROOF OF NECESSITY. Let $\{G_n\}$ denote the sequence of collections of neighborhoods given in Moore's Axiom 1_3. Since a Moore space is a semi-metric space [1], define the sequences $\{B_n\}$ and $\{J_n\}$ in a manner analogous to that in the proof of the necessity of Theorem 1 with the additional restriction that each member of J_n be a subset of an element of G_n, i.e., J_n refines G_n for each n. It follows that the sequences $\{J_n\}$ and $\{B_n\}$ satisfy Axiom Z_3.

PROOF OF SUFFICIENCY. Given that T satisfies Axiom Z_3, let G_n denote the collection of all elements B_i for all $i \geqq n$. It follows that the sequence $\{G_n\}$ satisfies Moore's Axiom 1_3.

THEOREM 3. *A necessary and sufficient condition for a topological space T to be metrizable is that T satisfy Axiom Z_4.*

2. **A correction.** The following counter-example shows that the argument for the statement *a Moore space is a semi-metric topological space* in [1, Theorem 6.2] is not correct.

Let the points of space be the points of the plane on or above the x-axis. Neighborhoods are of two types—interiors of circles above the x-axis and interiors of circles tangent to the x-axis from above together with the point of tangency. Let G_i denote the collection of all these neighborhoods whose diameters are less than $1/i$.

McAuley defined the distance between two points p and q as follows: *denote by n the least positive integer such that if* $g(p)$ *and* $g(q)$ *denote two neighborhoods in* G_n *containing the points p and q respectively, then* $g(p)\cdot g(q)=0$. *Define* $d(p, q)=1/n$. By this definition, a point on the x-axis is a distance limit point of the x-axis but is not a limit point of the x-axis.

It is not difficult to show that the following definition of distance is sufficient to show that a Moore space is a semi-metric topological space. *Let n denote the least natural number i such that if* $g(p)$ *and* $g(q)$ *are two neighborhoods in* G_i *containing p and q respectively, then* $g(p)$ *does not contain q and* $g(q)$ *does not contain p. Define* $d(p, q)=1/n$.

3. **Axiom A.** R. L. Moore's Axiom 1_3 is stated in terms of a sequence of collections of regions covering space. The following axiom is stated in terms of a point and a sequence of regions containing the point. The axiom is the same as Theorem 2 in [2] except for part (3).

AXIOM A. *If p denotes a point, there exists a sequence* $\{R_n(p)\}$ *where for each natural number n,* $R_n(p)$ *denotes a region containing p such that*:

(1) *p is the only point common to* $\{R_n(p)\}$,

(2) *for each n,* $R_{n+1}(p)$ *is a subset of* $R_n(p)$,

(3) *if R denotes a region containing p, then there exists an n such that if z denotes any point and* $R_n(z)$ *contains p,* $\overline{R}_n(z)$ *is a subset of R.*

THEOREM 4. *In the presence of Moore's Axiom* 0, *Axiom* 1_3 *is equivalent to Axiom* A.

PROOF. That Axiom 1_3 implies Axiom A is established by obtaining the sequence $\{R_n(p)\}$ as in the proof of Theorem 2 in [2] and showing that part (3) of Axiom A follows from part (3) of Axiom 1_3. To show that Axiom A implies Axiom 1_3, let G_n, for each n, denote the collection of all regions $R_i(p)$ for all natural numbers $i \geq n$ for all points p. It is easy to verify that the sequence $\{G_n\}$ satisfies Axiom 1_3.

REFERENCES

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