LOCAL VARIETIES AND ASYMPTOTIC EQUIVALENCE

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1. Let $\mathfrak{o}$ be a local domain and let $\mathfrak{a}$ be an ideal of $\mathfrak{o}$ that is primary for the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$. If α is an element that is superficial of degree s for $\mathfrak{a}$ (in the sense of [6, p. 22]) and $\alpha_1, \cdots, \alpha_t$ is a basis for $\mathfrak{a}^s$, let $\mathfrak{o}(\mathfrak{a}, \alpha) = \mathfrak{o}[\alpha_1/\alpha, \cdots, \alpha_t/\alpha]$. If $\mathfrak{m}'$ is a prime ideal of $\mathfrak{o}(\mathfrak{a}, \alpha)$ such that $\mathfrak{m}' \cap \mathfrak{o} = \mathfrak{m}$, then the quotient ring of $\mathfrak{o}(\mathfrak{a}, \alpha)$ at $\mathfrak{m}'$ is called an $\mathfrak{a}$-spot, and the totality of $\mathfrak{a}$-spots is called the variety $V(\mathfrak{a})$ of $\mathfrak{a}$. Let Ω be the set of all valuations of the quotient field F of $\mathfrak{o}$ that dominate $\mathfrak{o}$. It is shown that $V(\mathfrak{a})$ is a complete variety relative to Ω in the sense that each element v of Ω dominates a unique $\mathfrak{a}$-spot. If $\mathfrak{o}$ is analytically normal it is shown that for each $\mathfrak{m}$-primary ideal $\mathfrak{a}$ there is an integer g such that the integral closure $(\mathfrak{a}^g)_a$ of $\mathfrak{a}^g$ is such that all of its powers are integrally closed. Ideals with this property are called normal and $(\mathfrak{a}^g)_a$ is called a derived normal ideal of $\mathfrak{a}$ and denoted by $\mathfrak{a}_g$. If an ideal is normal, then its variety is normal and $V(\mathfrak{a}_g)$ is a normalization of $V(\mathfrak{a})$ in the sense that the integral closure in F of any spot P of $V(\mathfrak{a})$ is the intersection of a finite number of spots in $V(\mathfrak{a}_g)$ that dominate P. In §3 it is shown that there is a 1:1 correspondence between the algebraic points of the variety of the null-form ideal of $\mathfrak{a}$ (defined over the residue field k of $\mathfrak{o}$) and the set of $\mathfrak{a}$-spots of maximal rank. Hence it is not surprising that $V(\mathfrak{a})$ should have properties analogous to those of varieties over a field. In §4 some of these properties are elaborated. In particular, the usual relation of dominance between varieties $V(\mathfrak{a})$ and $V(\mathfrak{b})$ yields a partial order on the set of $\mathfrak{m}$-primary ideals that directs the set. If P and P' are $\mathfrak{a}$-spots such that P is a quotient ring of P' then we say that P' is a specialization of P. For each $\mathfrak{m}$-primary ideal $\mathfrak{a}$ there is a finite set $\{P_1, P_2, \cdots, P_s\}$ of $\mathfrak{a}$-spots such that each $\mathfrak{a}$-spot is a specialization of one of the P_i. An irreducible ideal is one whose spots are all specializations of a single spot P which is then called a general spot for the ideal. Finally, we show that the notion of asymptotic equivalence in the projective sense introduced by Samuel in [7] can be characterized in terms of local varieties. In fact, if $\mathfrak{a}$ and $\mathfrak{b}$ are asymptotically equivalent in the projective sense (in symbols: $\mathfrak{a} \barwedge \mathfrak{b}$) then $V(\mathfrak{a}) = V(\mathfrak{b})$, while if $\mathfrak{a}$ and $\mathfrak{b}$ are irreducible normal ideals such that $V(\mathfrak{a}) = V(\mathfrak{b})$ then it is shown that $\mathfrak{a} \barwedge \mathfrak{b}$.

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2. The proof of the existence of derived normal ideals is a paraphrase of Zariski's proof of the existence of normal varieties. First let $\mathfrak{o}$ be a complete local domain integrally closed in its quotient field F and assume that $\mathfrak{o}$ contains an infinite field k. Let $\mathfrak{a}=(a_1, a_2, \cdots, a_n)$ be an $\mathfrak{o}$-ideal and consider the restricted Rees ring

$$R = \mathfrak{o}[ta_1, ta_2, \cdots, ta_n],$$

where t is an indeterminate over F. An element θ of $F(t)$ that is integral over R must belong to $\mathfrak{o}[t]$ and if $\theta=\sum\theta_i t^i$, the components $\theta_i t^i$ of θ are integral over R since k is infinite. In particular, the element θ_i belongs to the integral closure $(\mathfrak{a}^i)_a$ of $\mathfrak{a}^i$. By a result of Rees [4], the fact that $\mathfrak{o}$ is analytically unramified implies that there is an integer h such that $(\mathfrak{a}^n)_a\subseteq\mathfrak{a}^{n-h}$ for all $n\geqq h$, so that if $i\geqq h$, we have $\theta_i t^i\in Rt^h$. Hence if $\overline{R}$ is the integral closure of R in $F(t)$, $\overline{R}\subseteq R+Rt+\cdots+Rt^h$ so that $\overline{R}$ is a finite R-module.[1] Moreover, since each integral element is the sum of homogeneous integral elements, we have $\overline{R}=\mathfrak{o}[z_1t^{e_1}, \cdots, z_st^{e_s}]$, where $z_i=a_i$, $e_i=1$ if $i=1, 2, \cdots, n$. Hence if g is a character of homogeneity (see [8]) associated with the integers $e_1, e_2, \cdots, e_s$, and if $T=\mathfrak{o}[w_1t^g, \cdots, w_Nt^g]$, is the ring generated over $\mathfrak{o}$ by the products of the $z_it^{e_i}$ of degree g, then T is integrally closed in its quotient field $F(t^g)$, and it follows readily that the ideal $\mathfrak{b}=\sum\mathfrak{o}w_i$ is normal. Since $\mathfrak{a}^g\subseteq\mathfrak{b}\subseteq(\mathfrak{a}^g)_a$ and since $\mathfrak{b}$ is integrally closed, it follows that $\mathfrak{b}=(\mathfrak{a}^g)_a$.

It is easily verified that if $\mathfrak{o}$ is a normal domain that is analytically normal and $\hat{\mathfrak{o}}$ is its completion, then for any primary ideal $\mathfrak{a}$ belonging to the maximal ideal of $\mathfrak{o}$, the operation of integral closure (a-operation) commutes with the operations of extension and contraction between $\mathfrak{o}$ and $\hat{\mathfrak{o}}$, so that we have proved

PROPOSITION 1. *If $\mathfrak{o}$ is a normal local domain that is analytically normal and contains an infinite field, and if $\mathfrak{a}$ is an $\mathfrak{o}$-ideal that is primary for the maximal ideal of $\mathfrak{o}$, then there exists an integer g such that $\mathfrak{a}_g=(\mathfrak{a}^g)_a$ is a normal ideal.*

3. Assume that $\mathfrak{o}$ is a normal domain with an infinite residue field k and fix a basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ for the $\mathfrak{m}$-primary ideal $\mathfrak{a}$. Let R and $\overline{R}$ denote the polynomial rings $\mathfrak{o}[X_1, \cdots, X_n]$ and $k[X_1, \cdots, X_n]$ respectively, where $X_1, X_2, \cdots, X_n$ are indeterminates. If $\phi\in R$, let $\bar{\phi}$ be the element of $\overline{R}$ obtained from ϕ by reading its coefficients modulo $\mathfrak{m}$. A form $\bar{\phi}_s(X)\in\overline{R}$ of degree s is called a null-form for $\mathfrak{a}$ if it is the image of a form $\phi_s(X)$ in R such that $\phi_s(\alpha)\in\mathfrak{a}^s\cdot\mathfrak{m}$, [3]. The

[1] The use of the Rees theorem here was suggested to us by M. Sakuma. It has resulted in a simplification of our original proof.

set of all null-forms for $\mathfrak{a}$ generates a homogeneous ideal $\mathfrak{n}(\mathfrak{a})$, and the variety of this ideal in the projective space $P_{n-1}(k)$ will be denoted by $N(\mathfrak{a})$.

Let $(c_1, c_2, \cdots, c_n)$ be an algebraic point of $N(\mathfrak{a})$ so that when defined, the ratios c_i/c_j are elements of $\bar{k}$, the algebraic closure of k. Assume for example that $c_1 \neq 0$ and let $\overline{G}_2(Y_2)$, $\overline{G}_3(Y_2, Y_3), \cdots, \overline{G}_n(Y_2, Y_3, \cdots, Y_n)$ be Zariski's canonical basis for the nonhomogeneous ideal of the point $(c_2/c_1, \cdots, c_n/c_1)$, [9]. If ν is the maximum degree of the polynomials $\overline{G}_i$, let $\bar{g}_i$ be the form of degree ν obtained from $\overline{G}_i$ by replacing each Y_j by X_j/X_1 and multiplying by X_1^ν. Finally, let g_i be a form of degree ν in R that reduces to $\bar{g}_i$ when taken modulo $\mathfrak{m}$, and denote the ideal $\mathfrak{m}\mathfrak{a}^\nu + \mathfrak{o}g_2(\alpha) + \cdots + \mathfrak{o}g_n(\alpha)$ by $\mathfrak{t}$. If $\theta \in \mathfrak{a}^{\nu g}$, there is a form $\phi(X)$ in R of degree νg such that $\theta - \phi(\alpha) \in \mathfrak{m}\mathfrak{a}^{\nu g}$. The representative $\bar{\phi}(X)$ of such a form is unique modulo $\mathfrak{n}(\mathfrak{a})$ and we denote it by $\theta_0(X)$.

LEMMA 3.1. *If $\theta \in \mathfrak{a}^{\nu g}$, then there is an integer μ such that $\alpha_1^{\nu\mu}\theta \in \mathfrak{t}\mathfrak{a}^{\nu(g-1+\mu)}$, if and only if $\theta_0(X)$ vanishes at $(c_1, c_2, \cdots, c_n)$.*

PROOF. Assume that $\theta_0(c) = 0$. The polynomial $\theta_0(1, Y_2, \cdots, Y_n)$ vanishes at $(c_2/c_1, \cdots, c_n/c_1)$ so that there exist polynomials $f_2(Y), \cdots, f_n(Y)$ such that $\theta_0(1, Y) = \Sigma f_i(Y)\overline{G}_i(Y)$. Let q be the least integer such that $(q+g-1)\nu$ is not less than the degree of any f_i. Then we find $X_1^{q\nu}\theta_0(X) = \Sigma f_i^*(X)\bar{g}_i(X)$, where f_i^* is the form of degree $(q+g-1)\nu$ obtained from f_i. Hence $\alpha_1^{q\nu}\phi(\alpha) \in \mathfrak{a}^{(q+g-1)\nu}\mathfrak{t}$, and $\alpha_1^{q\nu}\theta \in \mathfrak{a}^{\nu(g-1+q)}\mathfrak{t}$. Conversely, assume that $\alpha_1^{\nu q}\theta \in \mathfrak{a}^{\nu(g-1+q)}\mathfrak{t}$. Then there exist forms $F_2(X), \cdots, F_n(X)$ of degree $\nu(g-1+q)$ such that $\alpha_1^{\nu q}\theta - \Sigma F_i(\alpha)g_i(\alpha) \in \mathfrak{m}\mathfrak{a}^{\nu(g-1+q)}$. It follows that $X_1^{\nu q}\theta_0(X) - \Sigma \overline{F}_i(X)\bar{g}_i(X)$ is a null-form and hence vanishes at $(c_1, c_2, \cdots, c_n)$. Since $c_1 \neq 0$, this implies that $\theta_0(c) = 0$, q.e.d.

It follows from Lemma 3.1 that if $\theta_1 \in \mathfrak{a}^{\nu g}$ and $\theta_2 \in \mathfrak{a}^{\nu h}$ are such that $\alpha_1^{q\nu}\theta_1 \notin \mathfrak{a}^{(g+q-1)\nu}\mathfrak{t}$ and $\alpha_1^{q\nu}\theta_2 \notin \mathfrak{a}^{\nu(h+q-1)}\mathfrak{t}$ for all positive integers q, then also $\alpha_1^{q\nu}\theta_1\theta_2 \notin \mathfrak{a}^{\nu(h+g+q-1)}\mathfrak{t}$ for all positive integers q, so that the set $Z = \{\alpha/\beta;\ \alpha,\beta \in \mathfrak{a}^{\nu g},\ \alpha_1^{q\nu}\beta \notin \mathfrak{a}^{\nu(g+q-1)}\mathfrak{t},\ \forall q\}$ is a subring of the quotient field F of $\mathfrak{o}$. Moreover, the ring Z contains the ring

$$\mathfrak{o}^* = \mathfrak{o}[\omega_1/\alpha_1^\nu, \cdots, \omega_N/\alpha_1^\nu],$$

where $\omega_1, \omega_2, \cdots, \omega_N$ is the set of monomials of degree ν in $\alpha_1, \alpha_2, \cdots, \alpha_n$. (Hence in particular, if α_1 is superficial relative to $\mathfrak{a}$ then $\mathfrak{o}^*$ is the ring $\mathfrak{o}(\mathfrak{a}, \alpha_1^\nu)$.)

LEMMA 3.2. *If $u_1, u_2, \cdots, u_s$ is a basis for $\mathfrak{m}$, then the ideal generated in $\mathfrak{o}^*$, by $u_1, u_2, \cdots, u_s$ and $g_1(\alpha)/\alpha_1^\nu, \cdots, g_n(\alpha)/\alpha_1^\nu$ is a maximal ideal $\mathfrak{m}^*$ of $\mathfrak{o}^*$ such that $\mathfrak{m}^* \cap \mathfrak{o} = \mathfrak{m}$ and Z is the quotient ring of $\mathfrak{o}^*$ at $\mathfrak{m}^*$.*

PROOF. The ideal $\mathfrak{m}^*$ cannot be the unit ideal in $\mathfrak{o}^*$ for this would imply that for some integer q, $\alpha_1^{q\nu} \in \mathfrak{a}^{\nu(q-1)}\mathfrak{t}$, in contradiction with Lemma 3.1. Similarly, the ideal $\mathfrak{m}^*Z$ is not the unit ideal in Z. Moreover, an element α/β of Z is a nonunit of Z if and only if $\alpha_1^{\nu q}\alpha \in \mathfrak{a}^{\nu(q+\sigma-1)}\mathfrak{t}$ and hence $\alpha/\beta \in \mathfrak{m}^*Z$. Thus Z is a local ring and is clearly the quotient ring of $\mathfrak{o}^*$ at $\mathfrak{m}^*$. If $\mathfrak{o}_1 = \mathfrak{o}[\alpha_2/\alpha_1, \cdots, \alpha_n/\alpha_1]$ then $\mathfrak{o}^* = \mathfrak{o}_1$ and it is clear that $\mathfrak{o}_1/\mathfrak{m}^* \cong k(c_2/c_1, \cdots, c_n/c_1)$ so that $\mathfrak{m}^*$ is maximal, q.e.d.

Since k is infinite we can select a minimal base $\beta_1, \beta_2, \cdots, \beta_r$ of a minimal reduction of $\mathfrak{a}$ such that each β_i is superficial relative to $\mathfrak{a}$ (see [3 and 6]). It is assumed that the first r elements $\alpha_1, \cdots, \alpha_r$ of the $\mathfrak{a}$-basis above have these properties. Then at each algebraic point $C = (c_1, \cdots, c_r)$ of $N(\mathfrak{a})$ at least one of the first r coordinates c_i is different from zero. In fact, if $c_1 = c_2 = \cdots = c_r = 0$, then it is clear that $v(\alpha_i) > v(\mathfrak{a})$ $(i = 1, 2, \cdots, r)$ in any valuation v that dominates the local ring Z associated with C and this is not possible if $\alpha_1, \alpha_2, \cdots, \alpha_r$ generate a reduction of $\mathfrak{a}$, [3]. Hence it follows from Lemma 3.2 that the ring Z is a quotient ring of one of the rings $\mathfrak{o}(\mathfrak{a}, \alpha_1), \mathfrak{o}(\mathfrak{a}, \alpha_2), \cdots, \mathfrak{o}(\mathfrak{a}, \alpha_r)$ at a maximal ideal $\mathfrak{m}^*$ such that $\mathfrak{m}^* \cap \mathfrak{o} = \mathfrak{m}$. Thus Z is an $\mathfrak{a}$-spot of maximal rank and the function f defined by $f(C) = Z$ maps the algebraic points of $N(\mathfrak{a})$ into the set of $\mathfrak{a}$-spots of maximal rank. Moreover, it is clear that if $f(C) = f(C')$ then C and C' have the same ideal in $k[X_1, \cdots, X_n]$, so that they are k-isomorphic points of $N(\mathfrak{a})$. Hence if k-isomorphic points are identified then f is injective.

LEMMA 3.3. *If an $\mathfrak{a}$-spot P contains one of the rings $\mathfrak{o}(\mathfrak{a}, \alpha_i)$, $i = 1, 2, \cdots, r$, say $\mathfrak{o}(\mathfrak{a}, \alpha_1)$, then P is a quotient ring of $\mathfrak{o}(\mathfrak{a}, \alpha_1)$. If $v \in \Omega$ is such that its valuation ring dominates P and if $v(\alpha_i) = v(\mathfrak{a})$, then $\mathfrak{o}(\mathfrak{a}, \alpha_i) \subseteq P$.*

PROOF. By definition P is a quotient ring of a ring $\mathfrak{o}(\mathfrak{a}^\nu, \alpha)$ and hence $\alpha_1^\nu/\alpha \in P$. However, $\mathfrak{o}(\mathfrak{a}, \alpha_1) \subseteq P$ so that $\alpha/\alpha_1^\nu \in P$, α/α_1^ν is a unit of P and the first assertion of the lemma follows immediately. If say $v(\alpha_1) = v(\mathfrak{a})$ then $v(\alpha_1^\nu) = v(\mathfrak{a}^\nu) = v(\alpha)$ and hence α_1^ν/α is a unit in P. Hence $\alpha_j/\alpha_1 \in P$, and thus $\mathfrak{o}(\mathfrak{a}, \alpha_1) \supseteq P$.

COROLLARY. *Any $\mathfrak{a}$-spot P is a quotient ring of one of the rings $\mathfrak{o}(\mathfrak{a}, \alpha_1), \mathfrak{o}(\mathfrak{a}, \alpha_2), \cdots, \mathfrak{o}(\mathfrak{a}, \alpha_r)$.*

If P is an $\mathfrak{a}$-spot of maximal rank we can assume P to be a quotient ring of say $\mathfrak{o}(\mathfrak{a}, \alpha_1)$ at a maximal ideal $\mathfrak{m}^*$ that lies over $\mathfrak{m}$. The $\mathfrak{m}^*$-residue c_j of the quotient α_j/α_1 is then algebraic over k $(j = 2, 3, \cdots, n)$ and the point $C = (1, c_2, \cdots, c_n)$ belongs to the variety $N(\mathfrak{a})$. If $f(C)=Z$, then Z is an $\mathfrak{a}$-spot that is a quotient ring of $\mathfrak{o}(\mathfrak{a}, \alpha_1)$ so that clearly $Z=P$. We have thus proved

PROPOSITION 2. *There is a* 1:1 *correspondence between the classes of k-equivalent algebraic points of the variety $N(\mathfrak{a})$ and the set of $\mathfrak{a}$-spots of maximal rank.*

We conclude this section with the remark that the variety $V(\mathfrak{a})$ is necessarily complete with respect to the set Ω of valuations. Indeed, if $v \in \Omega$ then for some $i \leqq r$, $v(\alpha_i)=v(\mathfrak{a})$ and $\mathfrak{o}(\mathfrak{a}, \alpha_i)$ is contained in the valuation ring R_v of v. If $\mathfrak{p}$ is the center of v in $\mathfrak{o}(\mathfrak{a}, \alpha_i)$ then v dominates the $\mathfrak{a}$-spot $\mathfrak{o}(\mathfrak{a}, \alpha_i)_{\mathfrak{p}}$.

4. If $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathfrak{m}$-primary ideals of the local domain $(\mathfrak{o}, \mathfrak{m})$ we say that $\mathfrak{a}$ dominates $\mathfrak{b}$ (in symbols: $\mathfrak{a} \geqq \mathfrak{b}$) if each $\mathfrak{a}$-spot dominates some $\mathfrak{b}$-spot or equivalently if the variety $V(\mathfrak{a})$ dominates $V(\mathfrak{b})$. If $\mathfrak{a} \geqq \mathfrak{b}$ and P is a $\mathfrak{b}$-spot, then there is an $\mathfrak{a}$-spot that dominates P. In fact, if v is an element of Ω that dominates P, then there is an $\mathfrak{a}$-spot Q such that v dominates Q. Since Q dominates a $\mathfrak{b}$-spot P', v must dominate P and P' so that by Lemma 3.3 these spots are quotient rings of one and the same ring $\mathfrak{o}(\mathfrak{a}, \alpha_i)$. It follows that $P=P'$ since they must both be quotient rings at the center $\mathfrak{p}$ of v in $\mathfrak{o}(\mathfrak{a}, \alpha_i)$. This argument shows, incidentally, that the $\mathfrak{a}$-spot dominated by a given element v of Ω is unique.[2]

If $(\mathfrak{o}, \mathfrak{m})$ satisfies the hypothesis of Proposition 1, then it is easy to see that if $\mathfrak{a}_g$ is a derived normal ideal of $\mathfrak{a}$ then $\mathfrak{a}_g \geqq \mathfrak{a}$. For this purpose we need a lemma similar to [2, Proposition 1].

LEMMA 4.1. *Let $\mathfrak{a}$ be normal and let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be a minimal base of a minimal reduction of $\mathfrak{a}$. If π is the prime field of $\mathfrak{o}$ and $\phi(X)$ is a nonzero form of degree t in $\pi[X_1, X_2, \cdots, X_r]$, then $\mathfrak{a}^n : \mathfrak{o}\phi(\alpha)=\mathfrak{a}^{n-t}$ for all $n \geqq t$.*

PROOF. Pass to the completion $\hat{\mathfrak{o}}$ of $\mathfrak{o}$, let k be a coefficient field of $\hat{\mathfrak{o}}$ and form the power series ring $Q=k\{\alpha_1, \alpha_2, \cdots, \alpha_r\}$. Then $\hat{\mathfrak{o}}$ is a finite Q-module and it is easy to see that the integral dependence of an element ω upon $\hat{\mathfrak{a}}^n (\hat{\mathfrak{a}}=\hat{\mathfrak{o}}\mathfrak{a})$ is put into evidence by the minimal equation of ω over Q (see [2, Lemma 1]). Hence if $\phi \in \pi[X]$ and $\phi(\alpha)\omega \in \mathfrak{a}^n$ then ω must depend integrally upon $\mathfrak{a}^{n-t}$. In fact, if a_i is the coefficient of ω^{s-i} in the minimal polynomial of ω over Q, then $\phi(\alpha)^i a_i$ is the corresponding coefficient in the minimal polynomial of $\phi(\alpha)\omega$, and if $A=\mathfrak{a}\cap Q$, the fact that $\phi^i a_i \in A^{ni}$ implies that $a_i \in A^{ni-ti}$, since Q is regular. Since $\mathfrak{a}$ is normal, $\omega \in \mathfrak{a}^{n-t}$, q.e.d.

[2] The referee has kindly pointed out to us that in view of this fact and the Corollary to Lemma 3.3, the local variety $V(\mathfrak{a})$ is a complete (proper) schema in the sense of Grothendieck. See [1, Ch. II, §7].

Now let $\alpha_1, \cdots, \alpha_r$ be a minimal base of a minimal reduction of $\mathfrak{a}$ and let $\beta_i = \alpha_i^g$. Then $\beta_1, \cdots, \beta_r$ is a minimal base of a minimal reduction of $\mathfrak{a}^g$ and hence also of $\mathfrak{a}_g$. By Lemma 4.1 each β_i is superficial relative to $\mathfrak{a}_g$. Moreover, $\mathfrak{o}(\mathfrak{a}_g, \beta_i)$ is the integral closure of $\mathfrak{o}(\mathfrak{a}^g, \beta_i)$ in its quotient field. Hence each spot of $\mathfrak{a}_g$ dominates a spot of $\mathfrak{a}$ and since $\mathfrak{o}(\mathfrak{a}_g, \beta_i)$ is a finite module over $\mathfrak{o}(\mathfrak{a}^g, \beta_i)$ the integral closure of an $\mathfrak{a}$-spot P is the intersection of a finite number of $\mathfrak{a}_g$-spots that dominate P. Thus $V(\mathfrak{a}_g)$ is a normalization of $V(\mathfrak{a})$.

As mentioned above, an $\mathfrak{a}$-spot P is a specialization of an $\mathfrak{a}$-spot Q if Q is a quotient ring of P. If $\mathfrak{p}_{i1}, \mathfrak{p}_{i2}, \cdots, \mathfrak{p}_{is_i}$ is the set of minimal prime ideals of the principal ideal $\alpha_i\mathfrak{o}(\mathfrak{a}, \alpha_i)$, and if $P_{i1}, P_{i2}, \cdots, P_{is_i}$ are the corresponding $\mathfrak{a}$-spots, then each $\mathfrak{a}$-spot P is a specialization of at least one of the spots P_{ij}. In fact, if $\mathfrak{p}$ is a prime ideal in $\mathfrak{o}(\mathfrak{a}, \alpha_i)$ such that $\mathfrak{p} \cap \mathfrak{o} = \mathfrak{m}$ then $\alpha_i \in \mathfrak{p}$ and hence $\mathfrak{p} \supseteq \mathfrak{p}_{ij}$ for some j. If $\mathfrak{a}$ is a normal ideal $\mathfrak{o}(\mathfrak{a}, \alpha_i)$ is integrally closed and each P_{ij} is the valuation ring of a discrete archimedean valuation of the quotient field F of $\mathfrak{o}$. These are precisely the valuation rings of the valuations that occur in the Rees representation of the homogeneous pseudo-valuation $\bar{v}_{\mathfrak{a}}$ associated with $\mathfrak{a}$ [5]. It is natural to say that an $\mathfrak{a}$-spot P is generic for $\mathfrak{a}$ if all $\mathfrak{a}$-spots are specializations of P and to call an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ irreducible if it admits a generic spot. An ideal that is irreducible in our sense need not be irreducible as an element of the ideal semi-group under the intersection composition. We will show in a later paper, however, that an irreducible ideal as defined here is either simple in the sense of Zariski [10, Appendix 5] or is a power of a simple ideal.

PROPOSITION 3. *Let* $(\mathfrak{o}, \mathfrak{m})$ *be a local domain which satisfies the conditions of Proposition* 1, *let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be normal* $\mathfrak{m}$-*primary ideals of* $\mathfrak{o}$, *and let* $\mathfrak{c}$ *be a derived normal ideal of the product* $\mathfrak{a}\cdot\mathfrak{b}$. *Then* $\mathfrak{c} \geqq \mathfrak{a}$, $\mathfrak{c} \geqq \mathfrak{b}$, *and if* $\mathfrak{n}$ *is any normal ideal that dominates both* $\mathfrak{a}$ *and* $\mathfrak{b}$ *then* $\mathfrak{n} \geqq \mathfrak{c}$.

PROOF. Let v_1, v_2, v_3 be the homogeneous pseudo-valuations associated with $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ in the sense of Rees [5]. Each v_i can be represented as a minimum of a finite set of valuations: $v_i(x) = \min\{v_{i1}(x)/e_{i1}, \cdots, v_{is_i}(x)/e_{is_i}\}$, $i = 1, 2, 3$, where e_{ij} is an integer and v_{ij} is an integer valued element of Ω. This representation holds for all x in $\mathfrak{o}$. Let W be the set of valuations v_{ij} $(i = 1, 2, 3; 1 \leqq j \leqq s_i)$. Let P be a $\mathfrak{c}$-spot and let v be an element of Ω such that R_v dominates P. Select elements $\alpha \in \mathfrak{a}$, $\beta \in \mathfrak{b}$ such that $v(\alpha) = v(\mathfrak{a})$, $v(\beta) = v(\mathfrak{b})$ and $v_{ij}(\alpha) = v_{ij}(\mathfrak{a})$, $v_{ij}(\beta) = v_{ij}(\mathfrak{b})$ for all $v_{ij} \in W$. (Since $\mathfrak{o}$ has an infinite residue field the existence of such elements is trivial.) Since $\mathfrak{a}$ is normal the conditions $v_{ij}(\alpha) = v_{ij}(\mathfrak{a})$ imply that α is superficial relative to $\mathfrak{a}$. In fact, $\alpha x \in \mathfrak{a}^n$ implies that $v_{ij}(x) \geqq (n-1)v_{ij}(\mathfrak{a})$, and hence

$v_1(x) \geqq n-1$. Since $\mathfrak{a}^{n-1}$ is integrally closed, $x \in \mathfrak{a}^{n-1}$. Similarly, β is superficial relative to $\mathfrak{b}$ and if g is such that $\mathfrak{c} = (\mathfrak{a}^g \cdot \mathfrak{b}^g)_a$, then $\alpha^g \cdot \beta^g$ is superficial relative to $\mathfrak{c}$. Moreover, since $v(\alpha^g \beta^g) = v(\mathfrak{c})$, the ring $\mathfrak{o}(\mathfrak{c}, \alpha^g\beta^g)$ is contained in R_v and P is a quotient ring of $\mathfrak{o}(\mathfrak{c}, \alpha^g\beta^g)$. At the same time each of the rings $\mathfrak{o}(\mathfrak{a}, \alpha)$ and $\mathfrak{o}(\mathfrak{b}, \beta)$ is a subring of $\mathfrak{o}(\mathfrak{c}, \alpha^g\beta^g)$, so that P dominates an $\mathfrak{a}$-spot and a $\mathfrak{b}$-spot. Since any integrally closed ring that contains both $\mathfrak{o}(\mathfrak{a}, \alpha)$ and $\mathfrak{o}(\mathfrak{b}, \beta)$ must also contain $\mathfrak{o}(\mathfrak{c}, \alpha^g\beta^g)$, the second assertion of the proposition is clear.

PROPOSITION 4. *Under the hypotheses of Proposition* 3, *if* $\mathfrak{a} \barwedge \mathfrak{b}$ *then* $V(\mathfrak{a}) = V(\mathfrak{b})$. *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are normal and irreducible, then* $V(\mathfrak{a}) = V(\mathfrak{b})$ *implies* $\mathfrak{a} \barwedge \mathfrak{b}$.

PROOF. Let the Samuel functions $l_{\mathfrak{a}}(\mathfrak{b})$ and $L_{\mathfrak{a}}(\mathfrak{b})$ (see [7]) have the common value p/q. Then $l_{\mathfrak{a}^p}(\mathfrak{b}^q)$ and $L_{\mathfrak{a}^p}(\mathfrak{b}^q)$ have the common value 1 and $\mathfrak{a}^p$ and $\mathfrak{b}^q$ are integrally equivalent. Since $\mathfrak{a}$ and $\mathfrak{b}$ are normal, $\mathfrak{a}^p = \mathfrak{b}^q$, and $V(\mathfrak{a}) = V(\mathfrak{b})$. Conversely, if $\mathfrak{a}$ and $\mathfrak{b}$ are irreducible the homogeneous pseudo-valuations $\bar{v}_{\mathfrak{a}}$, and $\bar{v}_{\mathfrak{b}}$ are valuations, and the equality $V(\mathfrak{a}) = V(\mathfrak{b})$ implies that these valuations have the same valuation ring. Hence there is an integer valued valuation v of the quotient field of $\mathfrak{o}$ such that $\bar{v}_{\mathfrak{a}}(x) = v(x)/e$ and $\bar{v}_{\mathfrak{b}}(x) = v(x)/f$ for all x in $\mathfrak{o}$. Here e and f are integers and $e = v(\mathfrak{a})$, $f = v(\mathfrak{b})$. It follows that $\mathfrak{a}^e$ and $\mathfrak{b}^f$ determine the same homogeneous pseudo-valuation and are therefore integrally equivalent and hence equal, q.e.d.

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