EMBEDDING NUMBERS FOR FINITE GROUPS

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This note is concerned with the following problem. Let $H$ denote a subgroup of a finite group $G$ and let $L$ denote a linear or one dimensional representation (i.e., a character) of $H$. We assume throughout that the field $F$ is algebraically closed and is either of characteristic 0 or of prime characteristic which does not divide the order of any groups under consideration. Let $G|L$ denote the corresponding induced representation of $G$. How many distinct (i.e., nonequivalent) irreducible representations appear in the decomposition of $G|L$ into irreducible parts? (This number is just the central intertwining number of $G\downarrow L$, which is denoted by $\mathfrak{s}(G|L)$.) More specifically, we are interested in determining an upper bound on the number of distinct irreducible representations which will appear, purely in terms of the way $H$ is embedded in $G$, and in terms which do not depend on the particular linear representation $L$ of $H$. Two such bounds come quickly to mind. The number of classes (of conjugates) of the super group $G$, which we denote $\{G:e\}$, is clearly an upper bound. Dimension considerations also give $[G:L]$ as an upper bound.

We now introduce a new group theoretic invariant which heuristically is a measure of the manner in which the classes of $G$ are distributed among the $H$-cosets of $G$.

**Definition.** Let $H$ be a (not necessarily normal) subgroup of a finite group $G$. For each normal subset $N$ of $G$, let $\phi_1(N)$ denote the number of classes (of conjugates) of $G$ contained in $N$. Let $\phi_2(N)$ denote the number of right $H$-cosets of $G$ which have nonzero intersection with $N$. Let $\phi(N) = \{G:e\} - \phi_1(N) + \phi_2(N)$. We then define the **embedding number** of $H$ in $G$, denoted by $(G:H)$, to be the minimum of the $\phi(N)$, as $N$ is taken over all normal subsets of $G$. We remark that a definition of $\phi_2$ using left cosets would yield the same value for $(G:H)$ since $N^{-1}$ intersects the same number of left cosets as $N$ does right cosets.

Taking $N = \{e\}$ where $e$ is the identity element of the group we have $(G:H) \leq \{G:e\}$. Taking $N = G$ we have $(G:H) \leq [G:H]$. If $H \neq G$, it is easy to verify that $(G:H) > 1$. If $H$ is a proper normal subgroup, then, taking $N = H$ we have $(G:H) < \{G:e\}$. In the case where $H$ is a normal subgroup of $G$, another number associated with the embedding of $H$ in $G$ is the number of classes in the factor group $G/H$. We call this the **class number** of $H$ in $G$ and denote it by $\{G:H\}$.

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PROPOSITION 1. If \( H \) is a normal subgroup of a finite group \( G \), then \( \{G: H\} \leq (G: H) \).

**Proof.** Suppose \( (G: H) = m \). Then there exists a normal subset, say \( N \) of \( G \), such that \( N \) contains \( n \) classes of \( G \) and intersects \( c \) \( H \)-cosets such that \( m = \{G: e\} - n + c \). Note that \( n \geq c \) since \( m = (G: H) \leq \{G: e\} \). Let \( \overline{N} \) denote the smallest normal subset of the factor group \( G/H \), containing the \( c \) \( H \)-cosets which have nonzero intersection with \( N \). Then the number of classes of \( G/H \) contained in \( \overline{N} \) is less than or equal to \( c \). Let \( N' = \{x: x \in G \text{ and } x \text{ is contained in some } H \text{-coset belonging to } \overline{N}\} \). Then \( N' \) is a normal subset of \( G \) such that \( N \subset N' \) and thus \( N' \) contains at least \( n \) classes of \( G \). Then \( (G - N') \) is a normal subset of \( G \) and contains at most \( \{G: e\} - n \) classes of \( G \). Thus \( (G/H - \overline{N}) \) contains at most \( \{G: e\} - n \) classes of \( G/H \). Further \( \overline{N} \) contains at most \( c \) classes. Hence \( G/H \) contains at most \( \{G: e\} - n + c = m \) classes. Hence \( \{G: H\} \leq (G: H) \).

Thus in general we have \( \{G: H\} \leq (G: H) \leq \lceil G: H \rceil \). If \( G/H \) is abelian this degenerates to \( \{G: H\} = \langle G: H \rangle = \lceil G: H \rceil \). Now that we have a relative idea of how this new “embedding number” compares with the group theoretic invariants usually associated with the embedding of \( H \) in \( G \), we proceed to show the significance of \( \langle G: H \rangle \) in the theory of monomial representations. We must first prove a preliminary result.

**Lemma.** Let \( H \) be a subgroup of a finite group \( G \) and let \( L \) denote a linear representation of \( H \), over the field \( F \). Let \( G \left\vert L \right\rangle \) denote the corresponding induced representation of \( G \). Let \( D_1, D_2, \ldots, D_{n+1} \) denote distinct classes of \( G \) and let \( S_i = \sum_{x \in D_i} (G \left\vert L \right\rangle)_x \), for \( i = 1, 2, \ldots, n+1 \). If these \( n+1 \) classes are completely contained in the union of \( n \) right \( H \)-cosets of \( G \), then the \( S_i \), \( i = 1, 2, \ldots, n+1 \), are linearly dependent over \( F \).

**Proof.** Index the right \( H \)-cosets of \( G \), \( \{H \sigma_j\}, j = 1, \ldots, k \), in such a way that \( D_i \subset \bigcup_{j=1}^{n+1} H \sigma_j \), for \( i = 1, 2, \ldots, n+1 \). Then \( \{\sigma_j^{-1}: j = 1, \ldots, k\} \) form a set of representatives of the left \( H \)-cosets of \( G \). By [1, Corollary to Theorem 3], it is sufficient to show that there exists \( \alpha_1, \alpha_2, \ldots, \alpha_{n+1} \) in \( F \), not all zero, such that \( \sum_{i=1}^{n+1} \alpha_i \sigma_j = 0 \) for \( j = 1, \ldots, k \), where \( \beta_i = \sum_{x \in \sigma_j^{-1} D_i \cap H} L_x \) and \( \beta_j = 0 \) if \( \sigma_j^{-1} D_i \cap H \) is empty. Since \( L \) is linear we have \( \beta_j \in F \). Consider the set of homogeneous linear equations

\[ \sum_{i=1}^{n+1} \beta_{ij} x_i = 0, \quad j = 1, \ldots, n. \]

This system has \( n \) equations and \( n+1 \) unknowns and thus has a non-
trivial solution, say $x_i = a_i \in F$. Hence $\sum_{j=1}^{n+1} a_j \beta_{ij} = 0$ for $j = 1, \ldots, n$.

By our indexing of the $H$-cosets we have that $\sigma_j^{-1}D_i \cap H = D_i \sigma_j^{-1} \cap H = (D_i \cap H_\sigma_j) \sigma_j^{-1}$ is empty (and thus $\beta_{ij} = 0$), for $j > n$ and $i = 1, \ldots, n+1$. Hence $\sum_{j=1}^{n+1} a_j \beta_{ij} = 0$ for $j = 1, \ldots, k$.

**Theorem.** Let $H$ denote a subgroup of a finite group $G$ and let $L$ be a linear representation of $H$. Then the number of distinct irreducible representations appearing in the decomposition of the induced representation $G|L$ is less than or equal to $(G: H)$.

**Proof.** There exists a normal subset $N$ of $G$ such that $(G: H) = n - m + \phi_2(N)$, where $n = \{G: e\}$ and $m = \phi_1(N)$. Let $C_1, C_2, \ldots, C_m$ denote the classes of $G$ which are contained in $N$ and let $C_{m+1}, C_{m+2}, \ldots, C_n$ denote the remaining classes of $G$. Let $S_i = \sum_{x \in C_i} (G|L)_x$, for $i = 1, 2, \ldots, n$. By the previous lemma there are at most $\phi(N)$ elements among the $S_i$, $i = 1, 2, \ldots, m$, which are linearly independent over the field $F$. Hence there are at most $n - m + \phi_2(N) = (G: H)$ linearly independent elements among the $S_i$, $1 \leq i \leq m$. By [1, Theorem 1], $\mathfrak{s}(G|L) \leq (G: H)$. That is to say, the number of distinct irreducible representations appearing in the decomposition of $G|L$ is less than or equal to $(G: H)$.

**Corollary.** Let $H$ denote an abelian subgroup of a finite group $G$. Then $\{G: e\} \leq (G: H)[H: e]$.

**Proof.** Let $L$ denote the regular representation of $H$. Then $\mathfrak{s}(L) = \{H: e\} = [H: e]$ and each irreducible representation appearing in the decomposition of $L$ is linear. Thus by the theorem $\mathfrak{s}(G|L) \leq (G: H)[H: e]$. But $G|L$ is the regular representation of $G$ and thus $\mathfrak{s}(G|L) = \{G: e\}$.

**Remark.** If $H$ is a normal subgroup of $G$ and $L$ is the one-dimensional identity representation of $H$, then $G|L$ contains exactly $\{G: H\}$ distinct irreducible representations of $G$. Indeed it is sufficient to note that $G|L$ is the composition of the natural projection of $G$ on $G/H$ and the regular representation of $G/H$. The following proposition gives a sufficient condition for $\{G: H\}$ to be an upper bound to the number of distinct irreducible representations appearing in the decomposition of $G|L$, where $L$ is any linear representation of $H$. The referee conjectures that $\{G: H\}$ is such an upper bound whenever $H$ is a normal subgroup of $G$.

**Proposition 2.** Suppose $H$ is a normal subgroup of $G$ such that each class of $H$ is also a class of $G$. Then for every linear representation $L$ of $H$, $G|L$ contains at most $\{G: H\}$ distinct irreducible representations.

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* We are indebted to the referee for this remark.
Proof. The projection of each class of $G$ onto $G/H$ is contained in a class of $G/H$. Suppose $D_1$ and $D_2$ are two classes of $G$ whose projections on $G/H$ are contained in the same class of $G/H$. Then the projections of $D_1$ and $D_2$ on $G/H$ have nonempty intersection. Thus there exists $x \in D_1$, $y \in D_2$ and $h \in H$ such that $x = hy$.

Note that under our hypothesis $(G \mid L)_{ghg^{-1}} = L_h I$ for all $g \in G$, where $I$ is the identity operator on $\mathfrak{X}(G \mid L)$. Indeed for all $g, z \in G$, and $f \in \mathfrak{X}(G \mid L)$ we have $(G \mid L)_{ghg^{-1}} f(z) = f(ghg^{-1}z) = (L_{ghg^{-1}}z)f(z) = L_h f(z)$, where we have used the fact that $L$ is constant on the classes of $H$.

Let $n_i$ denote the number of elements in the class $D_i$, for $i = 1, 2$. Then we have

$$S_1 = \sum_{z \in D_1} (G \mid L)_z = \frac{n_1}{[G : e]} \sum_{g \in G} (G \mid L)_{gghg^{-1}} = \frac{n_1}{[G : e]} \sum_{g \in G} (G \mid L)_{ghg^{-1}} = \frac{n_1}{[G : e]} L_h \sum_{g \in G} (G \mid L)_{ghg^{-1}} = \frac{n_1}{n_2} L_h \sum_{z \in D_2} (G \mid L)_z = \left(\frac{n_1}{n_2} L_h\right) S_2.$$ 

Thus $\{G : H\}$ is an upper bound for the number of linearly independent conjugate sums $S_i$ and thus also for the number of distinct irreducible representations appearing in the decomposition of $(G \mid L)$.

For the theorem to have significance it is necessary to show that $(G : H)$ is indeed a better upper bound than those already known, namely $\{G : e\}$ and $[G : H]$. Let $G$ be the symmetric group on 4 letters. Let $H$ denote the normal abelian subgroup of $G$ of order 4. Then all the numbers associated with the embedding of $H$ in $G$ are distinct. Indeed $[G : H] = 6$, $\{G : e\} = 5$, $(G : H) = 4$ and $\{G : H\} = 3$.

It would be interesting to know if the embedding number $(G : H)$ has any significance in any other context than in the theory of group representations which are induced from characters.

Reference


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