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UNIVERSITY OF BRITISH COLUMBIA AND
UNIVERSITY OF FLORIDA

CARDINALITY OF LEVEL SETS OF RADEMACHER SERIES WHOSE COEFFICIENTS FORM A GEOMETRIC PROGRESSION[1]

W. A. BEYER

1. **Introduction.** With $0<r<1$, put

$$\beta^*(\alpha, r) = \left\{x \,\middle|\, \sum_{i=1}^{\infty} r^i R_i(x) = \alpha;\ 0 < x \leqq 1\right\}$$

where $R_i(x)$ is the ith Rademacher function and $|\alpha| < \sum_{i=1}^{\infty} r^i$. This paper discusses the cardinality of the set $\beta^*(\alpha, r)$ [hereafter denoted by card $\beta^*(\alpha, r)$]. The only previous discussion known to the author is a remark of Lévy [4]. Denote $(\sqrt{5}-1)/2$ by δ. In [2] we have shown that if $1>r>\delta$, then the Hausdorff dimension of β^* is $\geqq 1/n$ where n is the least n_0 such that

$$n_0 > \{\log(2r-1) - \log(r^2+r-1)\}/(-\log r).$$

Note that as $r\to\delta+$, $n\to\infty$. Hence card $\beta^*(\alpha, r)=c$ (cardinal number of the continuum) for $1>r>\delta$. It is known that card $\beta^*(\alpha, r)\leqq 1$ for $0<r<1/2$; $\beta^*(\alpha, r)=1$ or 2 for $r=1/2$. This leaves the range $1/2<r\leqq\delta$ in question. The question is completely settled for $r=\delta$ by Theorem 1. The range $1/2<r<\delta$ is discussed briefly in §4.

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We denote the set of algebraic integers in the quadratic field $k(\sqrt{5})$ by H and the cardinal number of the integers by $\aleph_0$. Our principal result is

THEOREM 1. *If* $\alpha \in H$, *then* card $\beta^*(\alpha, \delta) = \aleph_0$. *If* $\alpha \notin H$, *then* card $\beta^*(\alpha, \delta) = c$.

Since $R_i(x) = 1 - 2\epsilon_i(x)$ where $\epsilon_i(x)$ is the ith digit of the (unique) nonterminating binary expansion of $x \in (0, 1]$ and since $\sum_{i=1}^{\infty} r^i R_i(x)$ is absolutely convergent, it is sufficient to consider the sets

$$\beta(\alpha, r) = \left\{ x \,\middle|\, \sum_{i=1}^{\infty} r^i \epsilon_i(x) = \alpha;\ 0 < x \leqq 1 \right\}$$

with $0 < \alpha < \sum_{i=1}^{\infty} r^i$.

2. Preliminaries.

DEFINITION 1. For $x \in (0, 1]$ and $0 < r^2 < 1/2$, $T_r(x)$ is the plane point:

$$\left[\frac{1 - r^2}{r} \sum_{i=1}^{\infty} \epsilon_{2i-1}(x) r^{2i-1},\ \frac{1 - r^2}{r} \sum_{i=1}^{\infty} \epsilon_{2i}(x) r^{2i-1} \right].$$

DEFINITION 2. For $0 < r^2 \leqq 1/2$, C_r^1 is the Cantor type perfect set of constant ratio r^2 of dissection formed in $[0, 1]$ (see [5]); i.e., $\xi_i = r^2$ for all i. We define $B(C_r^1)$ as the set of end points of the removed intervals plus the points 0 and 1. We put $I(C_r^1) = C_r^1 - B(C_r^1)$. We define C_r^2 as the Cartesian product $C_r^1 \times C_r^1$. We also define the quantity $B(C_r^2) = B(C_r^1) \times B(C_r^1)$.

LEMMA 1. *The right-hand end points in* $B(C_r^1)$ *have the form* $[(1-r^2)/r] \sum_{i=1}^{p} \epsilon_i r^{2i-1}$. *The left-hand end points in* $B(C_r^1)$ *have the form* $[(1-r^2)/r][\sum_{i=1}^{p-1} \epsilon_i r^{2i-1} + \sum_{i=p+1}^{\infty} r^{2i-1}]$. *The points of* $I(C_r^1)$ *have the form* $[(1-r^2)/r] \sum_{i=1}^{\infty} \epsilon_i r^{2i-1}$ *with an* ∞ *of the* ϵ_i *equal to a* 0 *and an* ∞ *of the* ϵ_i *equal to* 1. *Here* $\epsilon_i = 0, 1$.

PROOF. See [5].

We take as evident

LEMMA 1a. $T_r(\beta(\alpha, r)) \subseteq l_{\alpha,r} \cap C_r^2$ *where* $l_{\alpha,r}$ *is the line* $rx_1 + r^2 x_2 = (1-r^2)\alpha$, *where* x_1, x_2 *are the coordinates of a plane point.*

3. Proof of Theorem 1.

A sequence of integers $\{\epsilon_i\}$ $(i = 1, 2, \cdots)$ is called a representation of α if $\epsilon_i = 0$ or 1 and $\alpha = \sum_{i=1}^{\infty} \epsilon_i r^i$. We denote the sequence by $R(\alpha)$. $R(\alpha) = \{\epsilon_i\}$ is canonical if $\alpha - \sum_{i=1}^{n} \epsilon_i r^i < r^n$ for every n for which $\epsilon_n = 0$. We then write $\alpha = \epsilon_1 \epsilon_2 \cdots$. We denote such $R(\alpha)$ by $R_c(\alpha)$.

In the remainder of this section, we take $r=\delta$. Each α has a unique $R_c(\alpha)$. Bergman [1] notes that $R_c(\alpha)$ is finite (terminates in zeros) if and only if $\alpha\in H$. On noting that r satisfies $1=r+r^2$, $r=\sum_{i=1}^{\infty} r^{2i}$ and $1=\sum_{i=2}^{\infty} r^i$, we obtain

LEMMA 2. *$R(\alpha)$ is canonical if and only if it does not terminate in a sequence of "01" pairs and no "11" pairs occur anywhere after the first zero.*

REMARK. It follows from Theorem 2 of [3] that the Hausdorff dimension of the set of $x\in(0, 1)$ identified with canonical $R(\alpha)$ is $-\log_2 \delta$.

LEMMA 3. *If $(x_1, x_2)\in l_{\alpha,r}\cap C_r^2$, $x_1=\sum_{j=1}^{\infty} \epsilon_j^1 r^{2j-1}$, $x_2=\sum_{j=1}^{\infty} \epsilon_j^2 r^{2j-1}$, then $\{\epsilon_i\}$ $(i=1, 2, \cdots)$, where $\epsilon_{2i-1}=\epsilon_i^1$ $(i=1, 2, \cdots)$ and $\epsilon_{2i}=\epsilon_i^2$ $(i=1, 2, \cdots)$ is an $R(\alpha)$. We then say that $R(\alpha)$ is a form resulting from $rx_1+r^2x_2$.*

PROOF. $\sum_{i=1}^{\infty} \epsilon_i r^i=\sum_{i=1}^{\infty} \epsilon_{2i-1} r^{2i-1}+\sum_{i=1}^{\infty} \epsilon_{2i} r^{2i}=\sum_{i=1}^{\infty} \epsilon_i^1 r^{2i-1}+\sum_{i=1}^{\infty} \epsilon_i^2 r^{2i}=x_1+rx_2=\alpha$.

LEMMA 4. *If $\alpha\in H$ and $(x_1, x_2)\in l_{\alpha,r}\cap C_r^2$, then $(x_1, x_2)\in B(C_r^2)$.*

PROOF. Suppose $(x_1, x_2)\notin B(C_r^2)$. At least one of the x_1 or x_2 is in $I(C_r^1)$ and hence has the form $\sum_{j=1}^{\infty} \epsilon_j r^{2j-1}$ with an ∞ of the ϵ_j equal to 1 and an ∞ equal to 0.

Case I. Suppose x_1 and x_2 are both in $I(C_r^1)$. The $R(\alpha)$ form resulting from $rx_1+r^2x_2$ must have an ∞ of 1's, an ∞ of 0's, and cannot terminate in a sequence of "01" pairs. This $R(\alpha)$ must therefore contain either (Ia) an ∞ of "11" pairs and an ∞ of 0's or (Ib) an ∞ of "00" pairs and an ∞ of 1's.

In case (Ia), we have an ∞ of "011" triplets in $R(\alpha)$. Replace these triplets with "100" triplets to obtain a new $R(\alpha)$. Thus, we need consider only case (Ib). Suppose case (Ib) holds. In $R(\alpha)$, define blocks of digits $N(i)$ $(i=1, 2, \cdots)$ with $N(i)$ as the block commencing with the ith "00" pair and extending to, but not including, the $(i+1)$st "00" pair. In $N(i)$, replace any "011" triplet by a "100" triplet. Repeat the operation until $N(i)$ is exhausted of "011" triplets. The result is a new $R(\alpha)$ with an ∞ of "00" pairs and isolated 1's. This $R(\alpha)$ is, by Lemma 2, canonical. It has an ∞ of 1's. Hence its value cannot be α.

Case II. Suppose $x_1\in B(C_r^1)$ and $x_2\in I(C_r^1)$ and $x_1=\sum_{i=1}^{n} \epsilon_i^1 r^{2i-1}$ with $\epsilon_n^1=1$. Consider the form $R(\alpha)$ resulting from $rx_1+r^2x_2$. Let N be the block consisting of the first $2n$ digits in $R(\alpha)$. In N replace any "011" triplet by a "100" triplet. Repeat the operation until N is exhausted

of "011" triplets. The resulting $R(\alpha)$ is, by Lemma 2, canonical and contains an ∞ of 1's, and hence cannot represent α.

Suppose $x_1 = \sum_{i=1}^{n} \epsilon_i^1 r^{2i-1} + \sum_{i=n+2}^{\infty} r^{2i-1}$. A contradiction is obtained as in Case I.

Case III. Suppose $x_1 \in I(C_r^1)$ and $x_2 \in B(C_r^1)$. This case is similar to Case II. This completes the proof of Lemma 4.

COROLLARY TO LEMMA 4. *If* $\alpha \in H$, *then* $T_r(\beta(\alpha, r)) \subseteq B(C_r^2)$.

LEMMA 5. *If* $\alpha \in H$, *then* card $\beta(\alpha, r) \leqq \aleph_0$.

PROOF. From the fact that the mapping T_r is 1-1, the above corollary, and the countability of $B(C_r^2)$, we have

$$\text{card } \beta(\alpha, r) = \text{card } T_r(\beta(\alpha, r)) \leqq \text{card } B(C_r^2) = \aleph_0.$$

LEMMA 6. *α has exactly $\aleph_0$ representations or $c(=2^{\aleph_0})$ representations according as $R_c(\alpha)$ is finite or is infinite (i.e., does not terminate in 0's).*

PROOF. Suppose $R_c(\alpha)$ is finite. From Lemma 5, card $\beta(\alpha, r) \leqq \aleph_0$. The sequence of equalities $\cdots 100 \cdots = \cdots 0110 \cdots = \cdots 010110 \cdots = \cdots$ gives at least $\aleph_0$ of the $R(\alpha)$. The terminal $1000 \cdots$ can be replaced in each case by $\cdots 010101 \cdots$ to give $\aleph_0$ of the $R(\alpha)$ which are nonterminating. Hence card $\beta(\alpha, r) = \aleph_0$. Suppose $R_c(\alpha)$ is infinite. In $R_c(\alpha)$ are found an ∞ of triplets of the form "100." Each triplet can, independently of the others, be replaced by "011." This gives c of the $R(\alpha)$. This completes the proof.

Lemma 6 and the observation about the number of nonterminating $R(\alpha)$ in case $R_c(\alpha)$ is finite give Theorem 1.

We note that

$$\text{card } \beta(\alpha, r) = \text{card } T_r(\beta(\alpha, r)) = \text{card}(l_{\alpha,r} \cap C_r^2)$$

since (1) T_r is 1-1, (2) from Lemma 3 if $P \in l_{\alpha,r} \cap C_r^2$ then there exists $x \in \beta(\alpha, r)$ such that $T_r(x) = P$, and (3) $T_r(\beta(\alpha, r)) \subseteq l_{\alpha,r} \cap C_r^2$.

REMARK. If $\alpha \in H$, then any $R(\alpha)$ terminates in either 0's, 1's, or "01" pairs. We do not give the proof.

4. $1/2 < r < \delta$.

THEOREM 2. *If $1/2 < r < \delta$, there exists at least a countably infinite set of values of α for which $\beta(\alpha, r)$ is a single point.*

PROOF. Let

$$\alpha = \sum_{j=1}^{\infty} r^{2j} = 0101010 \cdots. \tag{1}$$

Suppose another representation of α (in terms of the given r and a 0, 1

sequence) is desired. Suppose $\epsilon_{2k}=1$ in (1) is changed to 0 and that this is the first change. But $\sum_{j=k}^{\infty} r^{2j+1}=(r^{2k+1})/(1-r^2)=[r/(1-r^2)]r^{2k}<r^{2k}$, since for $0<r<\delta$, $1-r^2>r$. Hence, the first change cannot be a 1 to a 0.

Now suppose $\epsilon_{2k+1}=0$ in (1) is changed to 1 and that this is the first change. But $\sum_{j=k+1}^{\infty} r^{2j}=(r^{2k+2})/(1-r^2)=[r/(1-r^2)]r^{2k+1}<r^{2k+1}$. Hence the first change cannot be a 0 to a 1. Therefore, α has a unique representation.

The set A of values of α of the form $\alpha_p=\sum_{j=p}^{\infty} r^{2j}$ $(p=1, 2, \cdots)$ and $\alpha_p'=\sum_{i=1}^{2p-1} r^i+\sum_{j=p}^{\infty} r^{2j}$ $(p=1, 2, \cdots)$ will also have a unique representation. This completes the proof.

Let r_n be the positive root of the equation $1=\sum_{i=1}^{n} r^i$.

THEOREM 3. *For $n>2$ there exist sets of numbers A_i $(i=1, 2, 3)$ with* card $A_i=\aleph_0$ *such that*

(i) *if* $\alpha\in A_1$, card $\beta(\alpha, r_n)=1$,

(ii) *if* $\alpha\in A_2$, card $\beta(\alpha, r_n)=\aleph_0$,

(iii) *if* $\alpha\in A_3$, card $\beta(\alpha, r_n)=c$ *and the Hausdorff dimension of* $\beta(\alpha, r_n)\geqq 1/(n+1)$.

PROOF. The set A_1 is the set A constructed in the proof of Theorem 2. Let

$$A_2=\left\{\sum_{i=m}^{\infty} r_n^i\right\} \qquad (m=1, 2, \cdots)$$

and

$$A_3=\left\{\sum_{i=1}^{m} r_n^i+\sum_{i=1}^{\infty} r_n^{m+i(n+1)}\right\} \qquad (m=1, 2, \cdots).$$

Suppose $\alpha\in A_3$. Each of the ∞ of $(n+1)$-tuples $1\,00\cdots 0$ [n zeros] in the canonical representation of α can be replaced independently by the $(n+1)$-tuple $0\,11\cdots 1$ [n ones] to give c representations of α. The statement about Hausdorff dimension follows from Lemma 2 of [2].

Suppose $\alpha\in A_2$. The sequence of equalities

$$\begin{aligned}
\underbrace{0\cdots 0}_{m-1}111\cdots &= \underbrace{0\cdots 0}_{m-2}1\underbrace{0\cdots 0}_{n}111\cdots \\
&= \underbrace{0\cdots 0}_{m-2}1\underbrace{0\cdots 0}_{n-1}1\underbrace{0\cdots 0}_{n}111\cdots \\
&= \cdots
\end{aligned}$$

yields $\aleph_0$ representations of α. Now represent $\alpha \in A_2$ as

$$(2) \qquad \underbrace{0 \cdots 0}_{m-2} 1 \underbrace{0 \cdots 0}_{n-1} 1 \underbrace{0 \cdots 0}_{n-1} 1 \cdots .$$

Another representation of α can be obtained only by changing a digit in (2). Suppose the first digit changed is ϵ_i and $\epsilon_i = 1$. Then all the 0's to the right of ϵ_i must be changed to 1. Suppose $\epsilon_i = 0$. There are not enough 1's to the right to change to 0 to compensate for this gain. Hence α has only $\aleph_0$ representations.

REMARK. Theorems 1, 2, and 3 discuss the cardinality of the points of intersection of the lines $l_{\alpha,r}$: $rx_1 + r^2x_2 = (1-r^2)\alpha$ with the plane set C_r^2 $(1/2 < r \leqq \delta)$. Theorem 2 states that if the set C_r^2 is "thin" enough, some of these lines intersect C_r^2 in one point only. Theorem 1 states that for $r = \delta$, the set C_r^2 is sufficiently "thick" so that every $l_{\alpha,r}$ intersects C_r^2 in at least $\aleph_0$ points. Results in [2] give information about the Hausdorff dimension of $l_{\alpha,r} \cap C_r^2$ for $1 > r > \delta$.

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LOS ALAMOS SCIENTIFIC LABORATORY, UNIVERSITY OF CALIFORNIA