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CARDINALITY OF LEVEL SETS OF RADEMACHER SERIES
WHOSE COEFFICIENTS FORM A GEOMETRIC PROGRESSION

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1. Introduction. With 0 < r < 1, put

$$\beta*(\alpha, r) = \left\{ x \left| \sum_{i=1}^{\infty} r^i R_i(x) = \alpha; 0 < x \leq 1 \right. \right\}$$

where $R_i(x)$ is the $i$th Rademacher function and $|\alpha| < \sum_{i=1}^{\infty} r^i$. This paper discusses the cardinality of the set $\beta*(\alpha, r)$ [hereafter denoted by card $\beta*(\alpha, r)$]. The only previous discussion known to the author is a remark of Levy [4]. Denote $(\sqrt{5} - 1)/2$ by $\delta$. In [2] we have shown that if $1 > r > \delta$, then the Hausdorff dimension of $\beta*$ is $\leq 1/n$ where $n$ is the least $n_0$ such that

$$n_0 > \left\{ \log (2r - 1) - \log(r^2 + r - 1) \right\}/(-\log r).$$

Note that as $r \to \delta +$, $n \to \infty$. Hence card $\beta*(\alpha, r) = c$ (cardinal number of the continuum) for $1 > r > \delta$. It is known that card $\beta*(\alpha, r) \leq 1$ for $0 < r < 1/2$; $\beta*(\alpha, r) = 1$ or 2 for $r = 1/2$. This leaves the range $1/2 < r \leq \delta$ in question. The question is completely settled for $r = \delta$ by Theorem 1. The range $1/2 < r < \delta$ is discussed briefly in §4.

Presented to the Society, January 22, 1959 under the title Rademacher series with geometric coefficients; received by the editors April 4, 1961.

1 The work reported in this document was performed by Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology with the joint support of the U. S. Army, Navy, and Air Force under Air Force Contract AF 19(604)-5200.
We denote the set of algebraic integers in the quadratic field \( k(\sqrt{5}) \) by \( H \) and the cardinal number of the integers by \( \aleph_0 \). Our principal result is

**Theorem 1.** If \( \alpha \in H \), then \( \operatorname{card} \beta^*(\alpha, \delta) = \aleph_0 \). If \( \alpha \notin H \), then \( \operatorname{card} \beta^*(\alpha, \delta) = \mathfrak{c} \).

Since \( R_i(x) = 1 - 2\epsilon_i(x) \) where \( \epsilon_i(x) \) is the \( i \)th digit of the (unique) nonterminating binary expansion of \( x \in (0, 1] \) and since \( \sum_{i=1}^{\infty} r^i R_i(x) \) is absolutely convergent, it is sufficient to consider the sets

\[
\beta(\alpha, r) = \left\{ x \left| \sum_{i=1}^{\infty} r^i \epsilon_i(x) = \alpha; 0 < x \leq 1 \right. \right\}
\]

with \( 0 < \alpha < \sum_{i=1}^{\infty} r^i \).

2. Preliminaries.

**Definition 1.** For \( x \in (0, 1] \) and \( 0 < r^2 < 1/2 \), \( T_r(x) \) is the plane point:

\[
\left[ \frac{1 - r^2}{r} \sum_{i=1}^{\infty} \epsilon_{2i-1}(x) r^{2i-1}, \frac{1 - r^2}{r} \sum_{i=1}^{\infty} \epsilon_{2i}(x) r^{2i-1} \right].
\]

**Definition 2.** For \( 0 < r^2 \leq 1/2 \), \( C_r \) is the Cantor type perfect set of constant ratio \( r^2 \) of dissection formed in \([0, 1]\) (see [5]); i.e., \( \xi_i = r^i \) for all \( i \). We define \( B(C_r) \) as the set of end points of the removed intervals plus the points 0 and 1. We put \( I(C_r) = C_r - B(C_r) \). We define \( C_r^2 \) as the Cartesian product \( C_r \times C_r \). We also define the quantity \( B(C_r^2) = B(C_r) \times B(C_r) \).

**Lemma 1.** The right-hand end points in \( B(C_r) \) have the form \( [(1-r^2)/r] \sum_{i=1}^{\infty} \epsilon_i r^{2i-1} \). The left-hand end points in \( B(C_r) \) have the form \( [(1-r^2)/r] \sum_{i=1}^{\infty} \epsilon_i r^{2i-1} + \sum_{i=p+1}^{\infty} r^{2i-1} \). The points of \( I(C_r) \) have the form \( [(1-r^2)/r] \sum_{i=1}^{\infty} \epsilon_i r^{2i-1} \) with an \( \infty \) of the \( \epsilon_i \) equal to a 0 and an \( \infty \) of the \( \epsilon_i \) equal to 1. Here \( \epsilon_i = 0, 1 \).

**Proof.** See [5].

We take as evident

**Lemma 1a.** \( T_r(\beta(\alpha, r)) \subseteq l_{a, \alpha} \cap C_r^2 \) where \( l_{a, \alpha} \) is the line \( r x_1 + r^2 x_2 = (1-r^2) \alpha \), where \( x_1, x_2 \) are the coordinates of a plane point.

3. Proof of Theorem 1. A sequence of integers \( \{ \epsilon_i \} (i = 1, 2, \ldots) \) is called a representation of \( \alpha \) if \( \epsilon_i = 0 \) or 1 and \( \alpha = \sum_{i=1}^{\infty} \epsilon_i r^i \). We denote the sequence by \( R(\alpha) \). \( R(\alpha) = \{ \epsilon_i \} \) is canonical if \( \alpha = \sum_{i=1}^{\infty} \epsilon_i r^i < r^n \) for every \( n \) for which \( \epsilon_n = 0 \). We then write \( \alpha = \epsilon_1 \epsilon_2 \cdots \). We denote such \( R(\alpha) \) by \( R_c(\alpha) \).
In the remainder of this section, we take \( r = \delta \). Each \( \alpha \) has a unique \( R(\alpha) \). Bergman [1] notes that \( R(\alpha) \) is finite (terminates in zeros) if and only if \( \alpha \in H \). On noting that \( r \) satisfies \( 1 = r + r^3 \), \( r = \sum_{i=1}^{\infty} r^{2i} \) and \( 1 = \sum_{i=2}^{\infty} r^i \), we obtain

**Lemma 2.** \( R(\alpha) \) is canonical if and only if it does not terminate in a sequence of \( \text{"01" pairs} \) and \( \text{no \( \text{"11" pairs} \) occur anywhere after the first zero.}

**Remark.** It follows from Theorem 2 of [3] that the Hausdorff dimension of the set of \( x \in (0, 1) \) identified with canonical \( R(\alpha) \) is \(- \log_2 5\).

**Lemma 3.** If \((x_1, x_2) \in I_{x_1, x_2} \cap C_r^2 \), \( x_1 = \sum_{i=1}^{\infty} e_i r^{2i-1} \), \( x_2 = \sum_{i=1}^{\infty} e_i r^{2i-1} \), then \( \{e_i\} \) \((i = 1, 2, \ldots)\), where \( e_{2i-1} = e_i^1 \) \((i = 1, 2, \ldots)\) and \( e_{2i} = e_i^2 \) \((i = 1, 2, \ldots)\) is an \( R(\alpha) \). We then say that \( R(\alpha) \) is a form resulting from \( r x_1 + r^2 x_2 \).

**Proof.** \( \sum_{i=1}^{\infty} e_i r^i = \sum_{i=1}^{\infty} e_{2i-1} r^{2i-1} + \sum_{i=1}^{\infty} e_{2i} r^{2i} = \sum_{i=1}^{\infty} e_i r^{2i-1} + \sum_{i=1}^{\infty} e_i r^{2i} = x_1 + r x_2 = \alpha \).

**Lemma 4.** If \( \alpha \in H \) and \((x_1, x_2) \in I_{x_1, x_2} \cap C_r^2 \), then \((x_1, x_2) \in B(C_r^2)\).

**Proof.** Suppose \((x_1, x_2) \notin B(C_r^2)\). At least one of the \( x_1 \) or \( x_2 \) is in \( I(C_r^2) \) and hence has the form \( \sum_{i=1}^{\infty} e_i r^{2i-1} \) with an \( \infty \) of the \( e_i \) equal to 1 and an \( \infty \) equal to 0.

**Case I.** Suppose \( x_1 \) and \( x_2 \) are both in \( I(C_r^2) \). The \( R(\alpha) \) form resulting from \( r x_1 + r^2 x_2 \) must have an \( \infty \) of 1's, an \( \infty \) of 0's, and cannot terminate in a sequence of \( \text{"01" pairs} \). This \( R(\alpha) \) must therefore contain either (a) an \( \infty \) of \( \text{"11" pairs} \) and an \( \infty \) of 0's or (lb) an \( \infty \) of \( \text{"00" pairs} \) and an \( \infty \) of 1's.

In case (a), we have an \( \infty \) of \( \text{"011" triplets} \) in \( R(\alpha) \). Replace these triplets with \( \text{"100" triplets} \) to obtain a new \( R(\alpha) \). Thus, we need consider only case (b). Suppose case (b) holds. In \( R(\alpha) \), define blocks of digits \( N(i) \) \((i = 1, 2, \ldots)\) with \( N(i) \) as the block beginning with the \( i \)th \( \text{"00" pair} \) and extending to, but not including, the \((i+1)\)st \( \text{"00" pair} \). In \( N(i) \), replace any \( \text{"011" triplet} \) by a \( \text{"100" triplet} \). Repeat the operation until \( N(i) \) is exhausted of \( \text{"011" triplets} \). The result is a new \( R(\alpha) \) with an \( \infty \) of \( \text{"00" pairs} \) and isolated 1's. This \( R(\alpha) \) is, by Lemma 2, canonical. It has an \( \infty \) of 1's. Hence its value cannot be \( \alpha \).

**Case II.** Suppose \( x_1 \in B(C_r^2) \) and \( x_2 \in I(C_r^2) \) and \( x_1 = \sum_{i=1}^{\infty} e_i r^{2i-1} \) with \( e_1 = 1 \). Consider the form \( R(\alpha) \) resulting from \( r x_1 + r^2 x_2 \). Let \( N \) be the block consisting of the first \( 2n \) digits in \( R(\alpha) \). In \( N \) replace any \( \text{"011" triplet} \) by a \( \text{"100" triplet} \). Repeat the operation until \( N \) is exhausted
of "011" triplets. The resulting $R(\alpha)$ is, by Lemma 2, canonical and contains an $\infty$ of 1's, and hence cannot represent $\alpha$.

Suppose $x_1 = \sum_{i=1}^{n} \epsilon_i r^{2i-1} + \sum_{i=n+2}^{\infty} r^{2i-1}$. A contradiction is obtained as in Case I.

**Case III.** Suppose $x_1 \in I(C)$ and $x_2 \in B(C)$. This case is similar to Case II. This completes the proof of Lemma 4.

**Corollary to Lemma 4.** If $\alpha \in H$, then $T_r(\beta(\alpha, r)) \subseteq B(C)$.  

**Lemma 5.** If $\alpha \in H$, then $\text{card} \ \beta(\alpha, r) \leq \aleph_0$.

**Proof.** From the fact that the mapping $T_r$ is 1-1, the above corollary, and the countability of $B(C)$, we have

$$\text{card} \ \beta(\alpha, r) = \text{card} \ T_r(\beta(\alpha, r)) \leq \text{card} \ B(C) = \aleph_0.$$  

**Lemma 6.** $\alpha$ has exactly $\aleph_0$ representations or $c(=2^{\aleph_0})$ representations according as $R_c(\alpha)$ is finite or is infinite (i.e., does not terminate in 0's).

**Proof.** Suppose $R_c(\alpha)$ is finite. From Lemma 5, $\text{card} \ \beta(\alpha, r) \leq \aleph_0$. The sequence of equalities $\cdots 100 \cdots = 0110 \cdots = 010110 \cdots$ gives at least $\aleph_0$ of the $R(\alpha)$. The terminal $100 \cdots$ can be replaced in each case by $010101 \cdots$ to give $\aleph_0$ of the $R(\alpha)$ which are nonterminating. Hence $\text{card} \ \beta(\alpha, r) = \aleph_0$.

Suppose $R_c(\alpha)$ is infinite. In $R_c(\alpha)$ are found an $\infty$ of triplets of the form "100." Each triplet can, independently of the others, be replaced by "011." This gives $c$ of the $R(\alpha)$. This completes the proof.

Lemma 6 and the observation about the number of nonterminating $R(\alpha)$ in case $R_c(\alpha)$ is finite give Theorem 1.

We note that

$$\text{card} \ \beta(\alpha, r) = \text{card} \ T_r(\beta(\alpha, r)) = \text{card} \ (I_{a,r} \cap C^2)$$  

since (1) $T_r$ is 1-1, (2) from Lemma 3 if $P \subseteq I_{a,r} \cap C^2$ then there exists $x \in \beta(\alpha, r)$ such that $T_r(x) = P$, and (3) $T_r(\beta(\alpha, r)) \subseteq I_{a,r} \cap C^2$.

**Remark.** If $\alpha \in H$, then any $R(\alpha)$ terminates in either 0's, 1's, or "01" pairs. We do not give the proof.

4. $1/2 < r < \delta$.

**Theorem 2.** If $1/2 < r < \delta$, there exists at least a countably infinite set of values of $\alpha$ for which $\beta(\alpha, r)$ is a single point.

**Proof.** Let

$$\alpha = \sum_{j=1}^{\infty} r^{2j} = 0101010 \cdots$$

Suppose another representation of $\alpha$ (in terms of the given $r$ and a 0, 1
sequence) is desired. Suppose \( c_{2k} = 1 \) in (1) is changed to 0 and that this is the first change. But \( \sum_{j=k}^{n} r^{2j+1} = (r^{2k+1})/(1-r^2) = (r/(1-r^2))r^{2k} \) \(< r^{2k} \), since for \( 0 < r < \delta, 1-r^2 > r \). Hence, the first change cannot be a 1 to a 0.

Now suppose \( c_{2k+1} = 0 \) in (1) is changed to 1 and that this is the first change. But \( \sum_{j=k+1}^{n} r^{2j} = (r^{2k+2})/(1-r^2) = (r/(1-r^2))r^{2k+1} \). Hence the first change cannot be a 0 to a 1. Therefore, \( \alpha \) has a unique representation.

The set \( A \) of values of \( \alpha \) of the form \( \alpha_p = \sum_{j=p}^{n} r^{2j} \) (\( p = 1, 2, \ldots \)) and \( \alpha'_p = \sum_{j=p}^{n} r^j + \sum_{j=p}^{n} r^{2j} \) (\( p = 1, 2, \ldots \)) will also have a unique representation. This completes the proof.

Let \( r_n \) be the positive root of the equation \( 1 = 2^{n-1} r^2 \).

Theorem 3. For \( n > 2 \) there exist sets of numbers \( A_i \) (\( i = 1, 2, 3 \)) with card \( A_i = \aleph_0 \) such that

(i) if \( \alpha \in A_1 \), card \( \beta(\alpha, r_n) = 1 \),

(ii) if \( \alpha \in A_2 \), card \( \beta(\alpha, r_n) = \aleph_0 \),

(iii) if \( \alpha \in A_3 \), card \( \beta(\alpha, r_n) = \aleph_0 \) and the Hausdorff dimension of \( \beta(\alpha, r_n) \leq 1/(n+1) \).

Proof. The set \( A_1 \) is the set \( A \) constructed in the proof of Theorem 2. Let

\[
A_2 = \left\{ \sum_{i=m}^{\infty} r_i \right\} \quad (m = 1, 2, \ldots)
\]

and

\[
A_3 = \left\{ \sum_{i=1}^{m} r_i + \sum_{i=1}^{\infty} r_n^{m+i(n+1)} \right\} \quad (m = 1, 2, \ldots). \]

Suppose \( \alpha \in A_3 \). Each of the \( \infty \) of \( (n+1) \)-tuples 1 00 ... 0 \( [n \) zeros] in the canonical representation of \( \alpha \) can be replaced independently by the \( (n+1) \)-tuple 0 11 ... 1 \( [n \) ones] to give \( c \) representations of \( \alpha \). The statement about Hausdorff dimension follows from Lemma 2 of [2].

Suppose \( \alpha \in A_2 \). The sequence of equalities

\[
\begin{align*}
\begin{bmatrix}
0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
111 & \cdots \\
\end{bmatrix}
\begin{bmatrix}
m - 1 \\
\end{bmatrix}
&= \begin{bmatrix}
0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
m - 2 \\
\end{bmatrix}
\begin{bmatrix}
n \\
\end{bmatrix} \\
&= \begin{bmatrix}
0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
m - 2 \\
\end{bmatrix}
\begin{bmatrix}
n - 1 \\
\end{bmatrix} \\
&= \cdots
\end{align*}
\]
yields $\aleph_0$ representations of $\alpha$. Now represent $\alpha \in A_2$ as

$$m - 2 \ | \ n - 1 \ | \ n - 1$$

Another representation of $\alpha$ can be obtained only by changing a digit in (2). Suppose the first digit changed is $\epsilon_i$ and $\epsilon_i = 1$. Then all the 0's to the right of $\epsilon_i$ must be changed to 1. Suppose $\epsilon_i = 0$. There are not enough 1's to the right to change to 0 to compensate for this gain. Hence $\alpha$ has only $\aleph_0$ representations.

**Remark.** Theorems 1, 2, and 3 discuss the cardinality of the points of intersection of the lines $l_{a,r} : rx_1 + r^2x_2 = (1 - r^2)\alpha$ with the plane set $C_\delta (1/2 < r \leq \delta)$. Theorem 2 states that if the set $C_\delta$ is "thin" enough, some of these lines intersect $C_\delta$ in one point only. Theorem 1 states that for $r = \delta$, the set $C_\delta$ is sufficiently "thick" so that every $l_{a,r}$ intersects $C_\delta$ in at least $\aleph_0$ points. Results in [2] give information about the Hausdorff dimension of $l_{a,r} \cap C_\delta$ for $1 > r > \delta$.

**References**


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