1. Introduction. Let $l^2$ be the Hilbert space of square summable sequences $f = (f_0, f_1, f_2, \cdots)$. $l^2$ is isomorphic to the space $H^2$ of functions holomorphic in the unit disk $\Delta$ with square integrable boundary values, under the map

$$f \rightarrow F, \quad F(z) = \sum_{n=0}^{\infty} f_n z^n.$$

A Toeplitz operator is an $l^2$ linear operator $T$ to which corresponds a function $W$ on the unit circle $\Gamma$, such that under the isomorphism (1.1) we have for the inner product

$$\langle Tf, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{i\theta}) F(e^{i\phi}) G^*(e^{i\phi}) d\theta.
$$

Here $*$ denotes complex conjugation.

This work is concerned with the concrete spectral theory of Toeplitz operators that are associated with functions $W$ that satisfy the following two hypotheses:

(i) $W$ is real, bounded below, and absolutely integrable on $\Gamma$, but it is not equivalent to a constant function.

(ii) For each real $\lambda$ the set $\Gamma_\lambda = \{ \theta : W(\theta) \leq \lambda \}$ is, modulo a set of measure zero, an arc of the circle.

By concrete spectral theory we mean² that we exhibit an explicit sigma-finite measure $\rho$ on $(-\infty, \infty)$ and an explicit unitary correspondence $U : l^2 \rightarrow L^2(d\rho)$ such that $UTU^{-1} = M$, where $M$ is the multiplication operator on $L^2(d\rho)$ which sends $g(\lambda)$ into $\lambda g(\lambda)$. Hypothesis (i) implies that $T$ is bounded below, so its Friedrichs extension (again named $T$) is self-adjoint.

I am deeply indebted to the referee, who simplified the proof and helped clarify the exposition.

The results obtained are conveniently described in terms of the set of vectors $k(u) \in l^2$ defined for each $u \in \Delta$ by

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2 The author is a National Science Foundation fellow.

3 This description applies only to operators with simple spectrum. The hypotheses given guarantee that $T$ has simple spectrum, but this is not in general true for Toeplitz operators, cf. [6].

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(1.3) \[ k_n(u) = u^n, \quad k_0(0) = 1. \]

Under the correspondence (1.1) it follows that \( K(u; z) = (1 - uz)^{-1} \), and thus for each \( f \in l^2 \) we have

(1.4) \[ \langle f, k(u^*) \rangle = F(u). \]

Let \( E \) be the spectral measure of \( T \). In Theorem 3 we find that there is an absolutely continuous measure \( \rho \) on the line given by (3.3) and a collection of functions \( \Phi(u; \lambda) \) given by (3.2) such that for all \( u, v \in \Delta \) and each real Borel set \( A \)

(1.5) \[ \langle E(\Lambda)k(u), k(v) \rangle = \int_\Lambda \Phi(u; \lambda)\Phi^*(v, \lambda)d\rho(\lambda). \]

According to (2.7) and Lemma 1 the functions \( \Phi(u; \lambda) \) are for almost all \( \lambda \) holomorphic functions of \( u \in \Delta \) with the Maclaurin expansion

(1.6) \[ \Phi(u; \lambda) = \sum_{n=0}^{\infty} \phi_n(\lambda)u^n. \]

For fixed \( u \in \Delta, \Phi(u; \lambda) \in l^2(d\rho) \). Now the transformation \( k(u) \rightarrow \Phi(u; \lambda) \) defined on the set \( \mathcal{K} = \{ k(u) : u \in \Delta \} \) in \( l^2 \) with range in \( l^2(d\rho) \) preserves inner products as can be seen by taking \( \Lambda = (-\infty, \infty) \) in (1.5). Since \( \mathcal{K} \) is total in \( l^2 \), cf. (1.4), it follows that there exists a unique isometry \( U : l^2 \rightarrow l^2(d\rho) \) such that \( Uk(u) = \Phi(u; \lambda) \). This transformation has the explicit form below obtained from (1.3) and (1.6):

(1.7) \[ Uf = \sum_0^{\infty} f_n\phi_n. \]

We next note that (1.5) implies that for each real Borel set \( \Lambda \), \( U \) sends \( E(\Lambda)k(0) \) into the product of the indicator function of \( \Lambda \) and \( \Phi(0; \lambda) \). Since by (3.2) \( \Phi(0; \lambda) \) is almost everywhere nonzero, it follows that the range of \( U \) is \( l^2(d\rho) \). Thus \( U \) is a unitary mapping of \( l^2 \) onto \( l^2(d\rho) \).

As a corollary of the fact that (1.7) is a unitary equivalence we conclude that \( \{ \phi_n \} \) is a complete orthonormal set in \( l^2(d\rho) \). So we see that \( \Phi(u; \lambda) \) is a generating function for a complete orthonormal set of functions in \( l^2(d\rho) \). In the examples at the end we specify \( W \) so as to obtain certain Gegenbauer and Pollaczek polynomials.

2. Analysis of Toeplitz matrices. Suppose now that hypothesis (i) is satisfied, and set \( \lambda_0 = \text{ess inf } W \). It is known, [1], that whenever \( \lambda < \lambda_0 \) there is a factorization

(2.1) \[ W(\theta) - \lambda = |H_\lambda(e^{i\theta})|^2, \]
where $H_\lambda$ is an outer function in $H^2$, i.e., the set \( \{ z^*H_\lambda(z) \}_{n=0}^\infty \) is total in $H^2$. $H_\lambda$ is uniquely specified if we impose the normalization $H_\lambda(0) > 0$. The explicit formula is

\begin{equation}
H_\lambda(u) = \exp \int_{-\pi}^{\pi} \log (W(\theta) - \lambda) P^*(u^*, \theta) d\theta, \quad u \in \Delta
\end{equation}

where

\[ P(u, \theta) = \frac{1}{4\pi} (1 + ue^{i\theta})(1 - ue^{-i\theta})^{-1}. \]

Thus whenever $\lambda < \lambda_0$ and $T$ is given by (1.2) we may write $T - \lambda = S_\lambda^* S_\lambda$, where $S_\lambda$ is the operator which under the map (1.1) transforms into multiplication by $H_\lambda$, i.e.

\begin{equation}
S_\lambda f \leftrightarrow H_\lambda F.
\end{equation}

From (1.4) we easily compute that

\begin{equation}
S_\lambda^* k(u) = H_\lambda^*(u^*) k(u).
\end{equation}

Since $H_\lambda$ is an outer function, $S_\lambda$ and $S_\lambda^*$ have densely defined inverses. Hence $(T - \lambda)^{-1} = S_\lambda^{-1} S_\lambda^*$. Using (2.4) we see that

\begin{equation}
\langle (T - \lambda)^{-1} k(u), k(v) \rangle = H_\lambda^* (u^*) H_\lambda^{-1} (v) \langle k(u), k(v) \rangle.
\end{equation}

It is useful to rewrite this formula using (2.2) and the fact that $\langle k(u), k(v) \rangle = (1 - uv^*)^{-1}$. We obtain

\begin{equation}
\langle (T - \lambda)^{-1} k(u), k(v) \rangle = (1 - uv^*)^{-1} \exp - \int_{-\pi}^{\pi} \log (W(\theta) - \lambda) [P(u, \theta) + P^*(v, \theta)] d\theta.
\end{equation}

For $u, v \in \Delta$, the right hand side of (2.5') is holomorphic in the $\lambda$-plane cut along the real axis from $\lambda_0$ to $\infty$. Thus (2.5') provides an analytic continuation of the resolvent of $T$. We will apply the Stieltjes inversion formula

\begin{equation}
d(E(\lambda) k(u), k(v))/d\lambda = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \{\langle (T - \lambda - i\epsilon)^{-1} k(u), k(v) \rangle
- \langle (T - \lambda + i\epsilon)^{-1} k(u), k(v) \rangle \}
\end{equation}

to (2.5'), making use of the following

\textbf{Lemma 1.} For almost all real $\lambda$

\[ \int_{-\pi}^{\pi} \log | W(\theta) - \lambda | d\theta > - \infty. \]
Proof. Let $z = \lambda + i\epsilon$, $\epsilon > 0$, and consider $I(z) = \int_{-\pi}^{\pi} \log \left( W(\theta) - z \right) d\theta$. This equals

$$k \int_{-\pi}^{\pi} \log \left[ (W(\theta) - z)(1 + W(\theta)^2)^{-1/2} \right] d\theta,$$

where $k = (1/2) \int_{-\pi}^{\pi} \log \left[ 1 + W(\theta)^2 \right] d\theta$. Thus

$$I(z) = \int_{-\pi}^{\pi} \log \left[ (t - z)(1 + t^2)^{-1/2} \right] a(t) dt,$$

where $a$ is bounded and monotone. Upon partial integration we have

$$I(z) = \int_{-\pi}^{\pi} \left[ (t - z)^{-1} - (1 + t^2)^{-1/2} \right] a(t) dt.$$

Thus, $\lim_{\epsilon \to 0} I(z)$ exists a.e. and $\lim_{\epsilon \to 0} \Re I(z) = \lim_{\epsilon \to 0} \int_{-\pi}^{\pi} \log \left| W(\theta) - \lambda + i\epsilon \right| d\theta$ is finite a.e. The lemma now follows from this by monotone convergence.

The lemma allows us to conclude that

$$(2.7) \quad \Psi(u; \lambda) = \exp \left[ -\int_{-\pi}^{\pi} \log \left| W(\theta) - \lambda \right| P(u, \theta) d\theta \right]$$

defines for almost all $\lambda$ a holomorphic function of $u \in \Delta$. Let $\Gamma_\lambda = \{ \theta : W(\theta) \leq \lambda \}$. Evidently

$$\lim_{\epsilon \to 0} \log \left| W(\theta) - \lambda \pm i\epsilon \right| = \log \left| W(\theta) - \lambda \right| \quad \text{when} \quad \theta \in \Gamma_\lambda$$

$$= \log \left| W(\theta) - \lambda \right| \pm \pi i \quad \text{when} \quad \theta \in \Gamma_\lambda.$$

We can now calculate the right-hand side of (2.6) from (2.5'). The result is

**Theorem 1.** Let $T$ be a Toeplitz operator defined by (1.2), where $W$ satisfies hypothesis (i). Then the resolution of the identity $E(\lambda)$ of $T$ satisfies a.e.

$$d\langle E(\lambda)k(u), k(v) \rangle / d\lambda$$

$$= \pi^{-1} \Psi(u; \lambda) \Psi^*(v; \lambda) (1 - uv^*)^{-1} \sin \left\{ \pi \int_{\Gamma_\lambda} \left[ P(u, \theta) + P^*(v, \theta) \right] d\theta \right\}.$$

Theorem 1 describes the absolutely continuous part of $T$. That this is a complete description follows from

**Theorem 2.** Let $T$ be a Toeplitz operator defined by (1.2) where $W$ satisfies hypothesis (i). Then the spectral resolution of $T$ is weakly absolutely continuous with respect to Lebesgue measure.

Proof. See [6]. Theorem 2 makes use of the assumption that $W$ is not equivalent to a constant. The two theorems combined show that the spectrum of $T$ is purely continuous and consists of the closed
interval \([\text{ess inf } W, \text{ess sup } W]\). This result is known, cf. [3 and 4]. What remains to be done is to exploit Theorem 1. This will require the use of hypothesis (ii).

3. Concrete spectral theory. We shall need the simple computational

**Lemma 2.** If \(0 \leq b - a \leq 2\pi\),

\[
\int_a^b P(u, \theta) d\theta = \frac{1}{4\pi} (b - a) + \frac{1}{2\pi i} \log[(1 - u e^{i\alpha})(1 - u e^{i\beta})^{-1}].
\]

Hypothesis (i) says that \(\Gamma_\lambda = \{a(\lambda) \leq \theta \leq b(\lambda)\}\) where \(0 \leq b(\lambda) - a(\lambda) \leq 2\pi\). From Lemma 2 we have

\[
\pi \int_{\Gamma_\lambda} [P(u, \theta) + P^*(v, \theta)] d\theta
\]

\[
= \frac{1}{2} (b - a) + (2i)^{-1} \log \left[(1 - u e^{i\alpha})(1 - v^* e^{-i\alpha})(1 - u e^{i\beta})^{-1}(1 - v^* e^{-i\beta})^{-1}\right]
\]

where \(a = a(\lambda)\) and \(b = b(\lambda)\). The main theorem is

**Theorem 3.** Let \(T\) be a Toeplitz operator defined by (1.2) where \(W\) satisfies hypotheses (i) and (ii). Then in the spectral decomposition of \(T\) we have for each real Borel set \(\Lambda\)

\[
\langle \mathcal{E}(\lambda) k(u), k(v) \rangle = \int_\Lambda \Phi(u; \lambda) \Phi^*(v; \lambda) d\rho(\lambda)
\]

where

\[
\Phi(u; \lambda) = \Psi(u; \lambda) (1 - u e^{i\alpha(\lambda)})^{-1/2} (1 - u e^{i\beta(\lambda)})^{-1/2}
\]

and

\[
d\rho(\lambda) = \pi^{-1} \sin \frac{1}{2} [b(\lambda) - a(\lambda)] d\lambda.
\]

**Proof.** With the aid of (3.1) the sine term in the formula of Theorem 1 can be calculated. We obtain

\[
d\langle \mathcal{E}(\lambda) k(u), k(v) \rangle / d\lambda = \Phi(u; \lambda) \Phi^*(v, \lambda) \rho'(\lambda)
\]

for almost all \(\lambda\) where \(\rho'(\lambda) = d\rho(\lambda)/d\lambda\). Theorem 2 asserts that the set function \(\langle \mathcal{E}(\cdot) k(u), k(v) \rangle\) is absolutely continuous, whence the assertion of Theorem 3 follows.

**Corollary 1.** The functions \(\Phi(u; \lambda)\) are for almost all \(\lambda\) holomorphic functions of \(u \in \Delta\) with the Maclaurin expansion (1.6). The mapping
U: $\{f_n\} \rightarrow \sum_0^\omega f_n \phi_n$ is a unitary transformation of $l^2$ onto $L^2(d\rho)$ such that $UTU^{-1} = M$, where $M: g(\lambda) \rightarrow \lambda g(\lambda)$.

Proof. See the introduction.

4. Examples. The results obtained are:

Example 1. $W(\theta) = \cos \theta$.

$$\rho'(\lambda) = \pi^{-1}(1 - \lambda^2)^{1/2}, \quad |\lambda| < 1$$
$$= 0, \quad |\lambda| \geq 1.$$

$$\Phi(u; \lambda) = 2^{1/2}(1 - 2\lambda u + u^2)^{-1},$$
$$\phi_n(\lambda) = 2^{1/2} C_n^{(1)}(\lambda), \quad \text{where } C_n^{(1)}(\lambda) \text{ is the } n\text{th Gegenbauer polynomial of order } 1, \text{ cf. [2, p. 174].}$$

Example 2. $W(\theta) = \sin \theta$. $\rho'(\lambda)$ is as in Example 1.

$$\Phi(u; \lambda) = 2^{1/2}(1 - 2i\lambda u - u^2)^{-1},$$
$$\phi_n(\lambda) = 2^{1/2}(-i)^n C_n^{(1)}(\lambda).$$

Example 3. $W(\theta) = 1$ if $|\theta| < c$; $= 0$ otherwise. Here $0 < c < \pi$.

$$\rho'(\lambda) = \frac{1}{\pi} \sin c \text{ if } 0 < \lambda < 1, = 0 \text{ otherwise.}$$

Set $\beta = -1/2\pi \log (\lambda^{-1} - 1)$. Then

$$\Phi(u; \lambda) = e^{\beta} (1 + e^{-2\pi i})^{1/2} (1 - ue^{-i\beta} - i/2 - \beta)^{-1/2} - (1 - ue^{i\beta} - i/2 + \beta),$$
$$\phi_n(\lambda) = e^{\beta} (1 + e^{-2\pi i})^{1/2} P_n^{(1/2)}(\beta, c),$$

where $P_n^{(1/2)}(\beta, c)$ is the $n$th Pollaczek polynomial, cf. [5], of order $1/2$ for $(-\infty, \infty)$.

References


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