1. Introduction. All spaces referred to in this paper are assumed to be metric (or, more accurately, metrizable); metrics are denoted by symbols like $\rho$. A space $X$ is said to be of absolute Borel class $\alpha$ (where $\alpha$ is a countable ordinal) if, whenever $X$ is a subspace of a space $Y$, $X$ is Borel of class $\alpha$ in $Y$.\(^1\) Clearly, if $X$ is of absolute Borel class $\alpha$, then so is every space homeomorphic to $X$. For all but the simplest Borel classes, such spaces are characterized by the well-known theorem (see e.g., [6, p. 339]):

\begin{enumerate}
  \item $X$ is of absolute Borel class $\alpha$ if (and only if) $X$ is of Borel class $\alpha$ in some complete (metric) space.
\end{enumerate}

But this theorem applies only to the classes $G_\delta$ and beyond ($F_{\sigma\delta}$, $G_{\delta\sigma}$, etc., i.e., $\alpha \geq 2$). The only nontrivial case remaining is that of the absolute $F_\sigma$ spaces. It is well known (and elementary) that a separable space is an absolute $F_\sigma$ if and only if it is $\sigma$-compact (i.e., the union of countably many compact spaces). For nonseparable spaces this fails (e.g., an uncountable discrete set is an absolute $F_\sigma$ but not $\sigma$-compact); a valid generalization (Theorem 2 below) is the main object of this paper. We shall also characterize the spaces which are absolutely the difference between two closed (or, equivalently, open) sets; here the known result for separable spaces needs no alteration in the general case (Theorem 1), though the proof does. The more complicated types—e.g., the spaces which are absolutely the difference between two $F_\sigma$ (or, equivalently, $G_\delta$) sets—are covered by the standard theory [6, p. 339]; the analogue of (1) applies to them. For completeness, we mention that the only absolutely open space is the empty set; the absolutely closed spaces are, of course, the compact spaces.

2. Absolute $F_\cap G$ spaces. We state as a lemma the following fundamental result of Hausdorff:

**Lemma 1.** Given a closed subspace $A$ of a (metric) space $X$, and given any metric $\rho$ on $A$ (compatible with the topology of $A$), there exists a metric $\rho^*$ on $X$ (compatible with the topology of $X$) which, restricted to $A$, coincides with $\rho$.

For proofs, see [2; 3; 5].

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\(^1\) For simplicity of statement, we are ignoring here the distinction between additive and multiplicative Borel classes.
Theorem 1. A space $X$ is absolutely the difference between two closed sets if and only if it is locally compact.

One implication is well known (and applies to any Hausdorff spaces, not necessarily metric). If $X$ is a locally compact subset of a space $Y$, then $X$ is open relative to $Y$ (see e.g., [4, p. 99]), and is thus the intersection of an open set and a closed set in $Y$—or, equivalently, is the difference between two closed (or open) sets.

Conversely, suppose $X$ is not locally compact. There is a point $a \in X$ such that, for all $\epsilon > 0$, the closure of the neighborhood $S(a, \epsilon) = \{x \in X, \rho(a, x)<\epsilon\}$ fails to be compact. Then $S(a, 1)$ contains a countably infinite discrete set $B_1 = \{b_{1n}\} (n=1, 2, \ldots)$ with no limit point in $X$. $B_1$ is clearly closed in $X$; further, we may assume $a \notin B_1$ (otherwise replace $B_1$ by $B_1 - (a)$). Thus, on writing $\delta_1 = \rho(a, B_1)$, we have $0 < \delta_1 \leq 1$. Similarly $S(a, \delta_1/2)$ contains an infinite discrete (closed) set $B_2 = \{b_{2n}\} (n=1, 2, \ldots)$ with $0 < \rho(a, B_2) = \delta_2 \leq \delta_1/2$. Proceeding in this way we obtain, for each $m=1, 2, \ldots$, a closed infinite discrete set $B_m = \{b_{mn}\} (n=1, 2, \ldots)$ such that $0 < \rho(a, B_m) = \delta_m \leq \delta_{m-1}/2 (m>1)$. Put $A = (a) \cup \bigcup B_m$; clearly $A$ is closed.

Now consider the subset $A^*$ of the plane consisting of the points $a^* = (0, 0)$, $b_{mn}^* = (1/m, 1/(m+n-1)) (m, n=1, 2, \ldots)$. The mapping $f: A \to A^*$ defined by $f(a) = a^*$, $f(b_{mn}) = b_{mn}^*$, is easily verified to be a homeomorphism of $A$ onto $A^*$. Thus, if $d$ denotes the Euclidean distance in the plane, and if we put $\rho'(x, y) = d(f(x), f(y))$, then $\rho'$ is a metric on $A$ (compatible with the topology of $A$). By Lemma 1 there is an extension $\rho^*$ of $\rho'$ to a metric on $X$. Let $Y$ denote the completion of $X$ in this metric.

For each fixed $m$, the sequence $\{b_{mn}\} (n=1, 2, \ldots)$ is a Cauchy sequence in $X$, in the metric $\rho^*$, and it therefore converges to some $b_m \in Y$. We extend $f$ to $f^*$ by defining $f^*(b_m) = \lim_{n \to \infty} b_{mn}^* = (1/m, 0)$ = $b_m^*$ say; since $f$ is an isometry, so is $f^*$. The points $b_m (m=1, 2, \ldots)$ are therefore all distinct, and converge to $a$ as $m \to \infty$.

Suppose now that $X$ is the difference between two closed sets in $Y$; thus $X = F \cap G$ where $F$ is closed and $G$ is open in $Y$. Thus, for each $m$, $b_m \in \overline{F} \subset F$. Again, $b_m \to a \in X \subset G$; hence, for some $M$, $b_M \in G$. That is, $b_M \in F \cap G = X$; and the sequence $b_{mn} (n=1, 2, \ldots)$ converges to $b_M$ in $X$. But this contradicts the choice of $B_M$ as a discrete subset of $X$.

3. Absolute $F_\sigma$ spaces. We recall that the weight of a space $X$ is the least cardinal of any basis of open sets of $X$. Since we are dealing with metric spaces only, it is also the least cardinal of any dense subset of $X$. 
Lemma 2. If every point of a (metric) space \( X \) has a neighborhood of weight \( \leq k \), \( X \) can be expressed as a union of disjoint open sets \( X_\lambda \), each of weight \( \leq k \).

When \( k = \aleph_0 \) this is a well-known theorem of Alexandroff [1, p. 300]; another proof is in [7]. Actually this special case is all we shall need; but it may be of interest to sketch a simple proof of the general result. We may of course assume \( k \) infinite. Being paracompact, \( X \) has a locally finite covering \( \{ U_\alpha \} \) by open sets, each of weight \( \leq k \). Write \( U_\alpha \sim U_\beta \) to mean that there is a finite sequence \( U_\alpha = U_\gamma_1, U_\gamma_2, \ldots, U_\gamma_n = U_\beta \), such that \( U_\gamma_i \cap U_\gamma_{i-1} \neq \emptyset \) \( (i = 1, 2, \ldots, n) \). (That is, \( U_\alpha \) and \( U_\beta \) belong to the same component of the nerve of the covering.) This defines an equivalence relation; suppose the distinct equivalence classes are \( \{ C_\lambda \} \). For each \( \lambda \), put \( X_\lambda = \bigcup \{ U_\alpha \mid U_\alpha \in C_\lambda \} \). The sets \( X_\lambda \) are disjoint, open and cover \( X \); and, because each \( C_\lambda \) is easily seen to have cardinal \( \leq k \), each \( X_\lambda \) has weight \( \leq k \).

Lemma 3. Let \( k \) be an infinite cardinal, not the sum of \( \aleph_0 \) smaller cardinals. If every nonempty open subset of a nonempty complete (metric) space \( Y \) has weight \( \geq k \), \( Y \) contains a closed subset homeomorphic to the product \( B(k) \) of \( \aleph_0 \) discrete sets, each of cardinal \( k \).

This is a slight generalization of [8, Lemma 3.1], and is proved by the same argument.

Lemma 4. A necessary and sufficient condition that \( X \) be an absolute \( F_\sigma \) is that, in every metric on \( X \) (compatible with its topology), \( X \) is \( \sigma \)-complete.

If \( \rho \) is any metric on an absolute \( F_\sigma \) space \( X \), then \( X \) is \( F_\sigma \) in its completion \( \bar{X} \) in the metric \( \rho \). Thus \( X = \bigcup X_n \) \( (n = 1, 2, \ldots) \), where \( X_n \) is closed in \( \bar{X} \) and therefore complete with respect to \( \rho \). Conversely, if the condition is satisfied, let \( X \) be a subspace of a space \( Y \) with metric \( \rho \). Then \( \rho \) restricted to \( X \) gives a metric on \( X \), for which \( X = \bigcup X_n \) with \( X_n \) complete. Hence \( X_n \) is closed, and \( X \) is \( F_\sigma \), in \( Y \).

Lemma 5. Every closed subspace of an absolute \( F_\sigma \) is an absolute \( F_\sigma \).

Let \( A \) be closed in an absolute \( F_\sigma \) set \( X \). By Lemma 1, every metric \( \rho \) on \( A \) can be extended to a metric \( \rho^* \) on \( X \). By Lemma 4, \( X = \bigcup X_n \) where \( X_n \) is complete in \( \rho^* \). Hence \( A = \bigcup (A \cap X_n) \) where \( A \cap X_n \) is complete in the metric \( \rho \). By Lemma 4 again, \( A \) is an absolute \( F_\sigma \).

* The conclusion is in fact true even when \( k \) is the sum of \( \aleph_0 \) smaller cardinals, providing \( k > \aleph_0 \); but this needs further argument. We shall need Lemma 3 only when \( k = \aleph_1 \).
Lemma 6. Every \( F_\alpha \) subspace of an absolute \( F_\alpha \) is an absolute \( F_\alpha \).

Let \( A = \bigcup A_n \) where \( A_n \) is closed in an absolute \( F_\alpha \) space \( X \). If \( A \subset Y \), Lemma 5 shows that each \( A_n \) is \( F_\alpha \) in \( Y \); say \( A_n = \bigcup F_{nm} \) \((m=1, 2, \cdots)\) where \( F_{nm} \) is closed in \( Y \). Thus \( A = \bigcup_{m,n} F_{nm} \) and is \( F_\alpha \) in \( Y \).

Lemma 7. If \( X \) is a complete absolute \( F_\alpha \), then \( X = \bigcup X_n \) where each \( X_n \) is a locally separable \( F_\alpha \) subset of \( X \) \((n=1, 2, \cdots)\).

Define \( K_0 = X \), \( U_0 = \emptyset \). When \( K_\beta \), \( U_\beta \) have been defined for all ordinals \( \beta < \alpha \), we proceed as follows.

(a) If \( \alpha \) has a predecessor, \( \alpha^- \), put \( U_\alpha = \text{union of all relatively open separable subsets of } K_{\alpha^-} \), \( K_\alpha = K_{\alpha^-} - U_\alpha \).

(b) If \( \alpha \) is a limit ordinal, put \( U_\alpha = \emptyset \), \( K_\alpha = \bigcap \{ K_\beta \mid \beta < \alpha \} \).

Clearly \( U_\alpha \) is open relative to \( K_{\alpha^-} \), in case (a); hence the sets \( K_\alpha \) form a decreasing transfinite sequence of closed subsets of \( X \). For some sufficiently large ordinal \( \alpha^* \) (not necessarily countable, of course), we have \( K_{\alpha^*} = K_{\alpha^*+1} = \cdots = K_\alpha \), say. We shall show that \( K = \emptyset \). In fact, since \( U_{\alpha^*+1} = \emptyset \) here, the construction shows that each nonempty relatively open subset of \( K \) must be nonseparable, i.e., of weight \( \geq \aleph_1 \). By Lemma 3, if \( K \neq \emptyset \), \( K \) has a closed subset homeomorphic to \( B(\aleph_1) \). This in turn has a closed subset \( J \) homeomorphic to \( B(\aleph_0) \)—i.e., to the space of irrational numbers. Being closed in \( X \), \( J \) must be an absolute \( F_\alpha \), by Lemma 5. But a well-known consequence of Baire’s theorem is that \( J \) is not \( F_\alpha \) in the real line. Hence \( K = \emptyset \).

In what follows, \( \alpha \) runs over all ordinals \( < \alpha^* \). Since \( K = \emptyset \), it readily follows that \( \bigcup U_\alpha = X \). Define, for \( n=1, 2, \cdots \), \( V_{an} = \{ x \mid x \in U_\alpha, \rho(x, K_\alpha) \geq 1/n \} \). (We make the convention that \( \rho(x, \emptyset) = \infty \).) Then \( V_{an} \) is the intersection of the closed set \( \{ x \mid \rho(x, K_\alpha) \geq 1/n \} \) with \( U_\alpha \), which in turn is either \( \emptyset \) or the intersection of \( K_{\alpha^-} \) and an open subset of \( X \). Hence \( V_{an} \) is \( F_\alpha \) in \( X \). Further, one easily verifies that \( \rho(V_{an}, V_{\beta n}) \geq 1/n \) if \( \alpha \neq \beta \). Thus, if we put \( X_n = \bigcup_a V_{an} \), \( X_n \) is also \( F_\alpha \) in \( X \). Clearly \( U_{X_n} = \bigcup_a V_{an} \) = \( U_{\alpha} = X \). All that remains to be proved is that \( X_n \) is locally separable. Now, given \( x \in X_n \), we have \( x \in V_{an} \) for some \( \alpha \), and then \( S(x, 1/n) \cap X_n \subset U_{\alpha} \). But \( U_{\alpha} \) is, by construction, a union of open separable subsets of \( K_{\alpha^-} \) (for \( U_{\alpha} \neq \emptyset \) here); hence, for some \( \varepsilon > 0 \), \( S(x, \varepsilon) \cap K_{\alpha^-} \) is separable. If \( \delta = \min(\varepsilon, 1/n) \), then \( S(x, \delta) \cap X_n \subset S(x, \delta) \cap U_{\alpha} \subset S(x, \varepsilon) \cap K_{\alpha^-} \), and is therefore also separable.

Lemma 8. If \( X \) is a locally separable absolute \( F_\alpha \), \( X \) is \( \sigma \)-locally compact.

By Lemma 2 we have \( X = \bigcup X_\lambda \) where the sets \( X_\lambda \) are disjoint, open
and separable. Each $X_\lambda$ can be imbedded in a copy $H_\lambda$ of the Hilbert cube, and we may assume that the sets $H_\lambda$ are disjoint and of diameter 1. We extend the metrics on the separate sets $H_\lambda$ to a metric $\rho$ on $H = \bigcup H_\lambda$ by taking $\rho(x, y) = 1$ whenever $x \in H_\lambda$, $y \in H_\mu$, and $\lambda \neq \mu$. There is now an obvious homeomorphism of $X$ onto a subset $X'$ of $H$, and we have $X' = \bigcup F'_n$ $(n = 1, 2, \cdots)$ where $F'_n$ is closed in $H$. Clearly $F'_n$ is locally compact; in fact, if $x \in F'_n \cap \bigcap H_\lambda$ and so is a compact neighborhood of $x$ in $F'_n$. Thus $X = \bigcup F_n$ where $F_n$ is locally compact.

**Theorem 2.** A necessary and sufficient condition for a (metric) space to be an absolute $F_\sigma$, is that it be $\sigma$-locally compact.

Suppose $X$ is an absolute $F_\sigma$. By Lemma 4, $X = \bigcup X_m$ $(m = 1, 2, \cdots)$ where $X_m$ is complete (in the metric on $X$), and so closed. By Lemma 5, $X_m$ is an absolute $F_\sigma$. Hence, by Lemma 7, $X_m = \bigcup X_{mn}$ $(n = 1, 2, \cdots)$ where $X_{mn}$ is locally separable and $F_\sigma$ in $X$. By Lemma 6, $X_{mn}$ is an absolute $F_\sigma$; by Lemma 8 it follows that $X_{mn} = \bigcup X_{mnp}$ $(p = 1, 2, \cdots)$ where $X_{mnp}$ is locally compact. Thus finally

$$X = \bigcup \{X_{mnp} \mid m, n, p = 1, 2, \cdots\},$$

proving $X$ $\sigma$-locally compact.

Conversely, suppose $X$ is a $\sigma$-locally compact subset of a space $Y$, say $X = \bigcup X_n$ $(n = 1, 2, \cdots)$ where $X_n$ is locally compact. By Theorem 1, $X_n$ is the difference between two closed subsets of $Y$, and is therefore $F_\sigma$ in $Y$; say $X_n = \bigcup F_{nm}$ $(m = 1, 2, \cdots)$ where $F_{nm}$ is closed in $Y$. Then $X = \bigcup F_{nm}$ and is $F_\sigma$ in $Y$.

**Remark.** Since every separable locally compact space is $\sigma$-compact, Theorem 2 includes the classical result that for a separable space to be an absolute $F_\sigma$, it is necessary and sufficient that it be $\sigma$-compact.

**References**


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