ON A THEOREM BY I. GLICKSBERG

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If $X$ is a compact space we denote by $C(X)$ the Banach space of all continuous complex valued functions on $X$ with respect to the norm $\|f\| = \sup_{x \in X} |f(x)|$. Grothendieck [2, Theorem 5] has shown that a bounded subset of $C(X)$ is compact in the weak topology if and only if it is compact in the topology of pointwise convergence on $X$.

Using an extension of this theorem I. Glicksberg [1, Theorem 1.2] recently proved

**Theorem 1.** Let $X$ and $Y$ be locally compact Hausdorff spaces, and $f$ a bounded complex function on $X \times Y$ which is separately continuous, i.e., for which all the maps

$$x \rightarrow f(x, y) \quad \text{and} \quad y \rightarrow f(x, y)$$

are continuous. Then for $\mu$ a bounded Radon measure on $Y$,

$$x \rightarrow \int f(x, y) \, d\mu(y)$$

is continuous.

The purpose of this note is to prove the following more general

**Theorem 2.** Let $X$ be a locally compact and $Y$ a locally compact Hausdorff space. Let $f$ be a complex function on $X \times Y$ which is separately continuous, and $\mu$ a Radon measure on $Y$. Suppose furthermore that $|f(x, y)| \leq g(y)$ for all $x \in X$ and almost all $y \in Y$ (i.e., for all $y \in Y$ except a $\mu$-null set), where $g \in L^1(\mu)$. Then

$$x \rightarrow \int f(x, y) \, d\mu(y)$$

is continuous.

**Proof.** The proof could be derived from Theorem 1. However, we prefer to derive it directly from Grothendieck's theorem by a slight modification of Glicksberg's proof.

Let $\epsilon$ be any positive number. Since $g$ is zero outside the union of a $\mu$-null set and a countable set of compact sets, and every compact set

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$K \subseteq Y$ contains a compact set $K_1$ such that $\mu(K - K_1)$ is arbitrarily small and the restriction of $g$ to $K_1$ is continuous, there exists a compact set $K \subseteq Y$ such that the restriction of $g$ to $K$ is continuous, and such that

$$\int_{Y - K} g(y) d\mu(y) < \epsilon.$$ 

We may furthermore assume that $K$ is such that

$$|f(x, y)| \leq g(y) \text{ for all } x \in X \text{ and all } y \in K.$$

Since $g$ is continuous on $K$, there exists a constant $M$ such that $|f(x, y)| \leq M$ for all $x \in X$ and all $y \in K$.

Let now $a$ be any point of $X$ and $(x_\alpha)$ be any generalized sequence (i.e., directed set) of points in $X$ which converges to $a$, and $V$ be a compact neighborhood of $a$. Denote by $f_\alpha$ the restriction of the mapping $y \mapsto f(x, y)$ to $K$. Then $A = \{f_\alpha\}, x \in V$, is a bounded subset of $C(K)$ which is compact in the topology of pointwise convergence on $X$, since the mapping $x \mapsto f_\alpha$ of $X$ into $C(K)$ endowed with the topology of pointwise convergence on $K$ is continuous. Hence by Grothendieck's theorem $A$ is compact in the weak topology of $C(K)$ and the two topologies agree on $A$ since they are comparable. It follows that $(f_\alpha), x_\alpha \in V$, converges weakly to $f_\alpha$. Hence there exists an $\alpha$ such that

$$\left| \int_K \left[ f(a, y) - f(x_\beta, y) \right] d\mu(y) \right| < \epsilon \quad \text{for } \beta > \alpha.$$ 

Therefore

$$\left| \int \left[ f(a, y) - f(x_\beta, y) \right] d\mu(y) \right|$$

$$\leq \left| \int_K \left[ f(a, y) - f(x_\beta, y) \right] d\mu(y) \right| + 2 \int_{Y - K} g(y) d\mu(y) < 3\epsilon \quad \text{for } \beta > \alpha,$$

which proves that the mapping $x \mapsto \int f(x, y) d\mu(y)$ is continuous at $a$.

References


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