# ON COMPLETE BERGMAN METRICS

SHOSHICHI KOBAYASHI[1]

1. In [3] we gave a sufficient condition for the Bergman metric to be complete. We shall give here a slightly modified condition for the completeness. To state our result more explicitly, we shall recall definitions given in [3].

2. Let $M$ be an $n$-dimensional complex manifold, $F$ the Hilbert space of holomorphic $n$-forms $f$ on $M$ such that

$$(-1)^{n^2/2}\int_M f\wedge\bar{f}<\infty.$$

Let $h_0, h_1, h_2, \cdots$ be an orthonormal basis for $F$. The kernel form $K$ of Bergman is defined by

$$K=\sum h_i\wedge\bar{h}_i.$$

(Strictly speaking, one should put $(-1)^{n^2/2}$ in front of $\sum$; but this is not essential in the following discussion.)

Suppose $F$ is ample in the following sense:

(A.1). For every $z$ in $M$, there exists an $f$ in $F$ which does not vanish at $z$.

(A.2). For every holomorphic vector $Z$ at $z$, there exists an $f$ in $F$ such that $f$ vanishes at $z$ and $Z(f^*)\neq 0$, where $f=f^*\,dz^1\wedge\cdots\wedge dz^n$ with respect to a local coordinate system $z^1, \cdots, z^n$ of $M$.

If $F$ satisfies the conditions (A.1) and (A.2), then the Bergman metric $ds^2$ is defined by

$$ds^2=\sum\frac{\partial^2\log K^*}{\partial z^\alpha\partial\bar{z}^\beta}dz^\alpha d\bar{z}^\beta$$

where $K=K^*\,dz^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^1\wedge\cdots\wedge d\bar{z}^n$:

If $M$ is a bounded domain in $C^n$, then $F$ is ample and the Bergman metric is defined; this is of course the case originally considered by Bergman [1].

3. Consider now the following additional condition

(C). For every infinite sequence $S$ of points of $M$ which has no adherent point in $M$ and for every $f$ in $F$, there exists a subsequence $S'$ of $S$ such that

$$\lim_{S'}(f\wedge\bar{f})/K=0.$$

Received by the editors June 7, 1961.

[1] This work has been supported by N.S.F. Grant 10375.

Then we know that

(i). A complex manifold $M$ satisfying (C) is complete with respect to the Bergman metric [3]. (I conjecture that the converse is true.)

(ii). A bounded domain in $C^n$ which is complete with respect to the Bergman metric is a domain of holomorphy. The converse is not true [2].

(iii). Every domain of holomorphy can be approximated by an increasing sequence of analytic polyhedrons.

(iv). Every bounded analytic polyhedron in $C^n$ satisfies (C) [3].

The above four statements show that the three concepts "holomorph-convexity," "metric completeness" and "(C)" are closely related to each other. Concerning (ii), it is not known whether a complex *manifold* which is complete with respect to the Bergman metric is necessarily holomorph-convex. It is also unknown whether (C) implies the holomorph-convexity for a manifold. For the proof of (ii), Bremermann makes use of the ambient space $C^n$ which is not available in the case of an abstract complex manifold. Let $M$ be a bounded domain of holomorphy in $C^n$ and let $A(M)$ be the intersection of all the domains of holomorphy $G$ containing the closure of $M$. According to Sommer-Mehring [4], the assumption $A(M)=M$ implies that the kernel function can not be continued outside $M$. It is very likely that $A(M)=M$ implies the completeness with respect to the Bergman metric.

4. We shall now consider the following condition

(C′). Let $F'$ be a (fixed) dense subset of the Hilbert space $F$. For every infinite sequence $S$ of points of $M$ which has no adherent point in $M$ and for every $f$ in $F'$, there exists a subsequence $S'$ of $S$ such that

$$\lim_{S'} (f \wedge \bar{f})/K = 0.$$

We shall prove

THEOREM. *If a complex manifold $M$ with Bergman metric satisfies* (C′) *(for some dense subset $F'$ of $F$), then $M$ is complete with respect to the Bergman metric.*

The proof is a slight modification of the argument in our previous paper [3, p. 284] and we shall use the same notations as in [3]. Let $H$ be the dual space of $F$ and $P(H)$ the projective space of complex 1-dimensional subspaces of $H$; the dimension of $P(H)$ is possibly infinite. In [3, (see pp. 280–282)], we defined a natural Kaehler metric $d\sigma^2$ on $P(H)$ and proved the metric completeness of $P(H)$. The natural imbedding $j: M \rightarrow P(H)$ defined in [3] is isometric in the sense of differential geometry, i.e., $j^*(d\sigma^2)=ds^2$. The distance between two points of $M$ (resp. $P(H)$) is the greatest lower bound of the lengths of the piecewise differentiable curves joining them in $M$ (resp. $P(H)$). It follows that, for every pair of points $z$ and $z'$ of $M$, the distance between $j(z)$ and $j(z')$ with respect to $d\sigma^2$ does not exceed the one between $z$ and $z'$ with respect to $ds^2$. Assuming that $M$ is not complete, let $S$ be a Cauchy sequence in $M$ which has no limit point in $M$. Then $j(S)$ is a Cauchy sequence in $P(H)$. By the completeness of $P(H)$, $j(S)$ has a limit point, say $x_0$, in $P(H)$. By a proper choice of basis in $H$, we may assume that $x_0$ is represented by a point $\zeta_0=(1, 0, 0, \cdots)$ of $H$. Take the dual basis $h_0, h_1, h_2, \cdots$ in $F$. Let $f$ be an element of $F'$. Then

$$f = \sum_{j=0}^{\infty} a_j h_j, \qquad a_j \in C.$$

For any $z$ in $M$, $j(z)$ is represented by the point of the unit sphere in $H$ whose homogeneous coordinates are given by

$$(h_0(z) \wedge \bar{h}_0(z)/K(z, \bar{z}), \qquad h_1(z) \wedge \bar{h}_1(z)/K(z, \bar{z}),$$
$$h_2(z) \wedge \bar{h}_2(z)/K(z, \bar{z}), \cdots).$$

Hence, $\lim_S (f\wedge\bar{f})/K=|a_0|^2$. Let $S'$ be any subsequence of $S$. Since $j(S')$ and $j(S)$ have the same limit point,

$$\lim_{S'} (f \wedge \bar{f})/K = |a_0|^2.$$

In order that the condition (C′) holds, $a_0$ must be zero. That would imply that $F'$ is orthogonal to $h_0$, contradicting the assumption that $F'$ is dense in $F$. Q.E.D.

COROLLARY. *Let $M$ be a bounded domain in $C^n$. If the polynomials are dense in $F$ and if the Bergman's kernel function goes to infinity at every boundary point of $M$, then $M$ is complete with respect to the Bergman metric.*

## BIBLIOGRAPHY

1. S. Bergman, *Ueber die Kernfunktion eines Bereiches und ihr Verhalten am Rande*, J. Reine Angew. Math. **169** (1933), 1–42; **172** (1935), 89–128.

2. H. Bremermann, *Holomorphic continuation of the kernel function and the Bergman metric in several complex variables*, Lectures on functions of a complex variable, University of Michigan Press, Michigan (1955), 349–383.

3. S. Kobayashi, *Geometry of bounded domains*, Trans. Amer. Math. Soc. **92** (1959), 267–290.

4. J. Mehring and F. Sommer, *Kernfunktion und Hüllenbildung in der Funktionentheorie mehrerer Veränderlichen*, Math. Ann. **131** (1956), 1–16.

STANFORD UNIVERSITY AND
UNIVERSITY OF BRITISH COLUMBIA