SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

SHORT PROOF OF A THEOREM OF WINTNER ON ASYMPTOTIC INTEGRATIONS OF THE ADIABATIC OSCILLATOR

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The theorem below on real functions was proved by Wintner [2] and generalized by Levinson [1]. “Solution" to an nth order system of differential equations will mean a set of n absolutely continuous functions satisfying the given system almost everywhere.

THEOREM. Let \( g > p > -1 \) (\( p \) a constant) be a function of bounded variation on \( \{ x : 0 \leq x < \infty \} \). The differential equation \(-y'' = (f')^2y\), where \( f' = \frac{d}{dx} \) and \( f = \int_0^x (1 + g)^{1/2} \), has solutions \( y_1 \) and \( y_2 \) such that \( y_1 \to \sin f \) and \( y_2 \to \cos f \) while \( y_1' \to f' \cos f \) and \( y_2' \to -f' \sin f \) as \( x \to \infty \).

Proof. Denote by \( J \) the set on which \( g \) is differentiable, and put \( F = f''/f' \). As suggested by the case where \( f' \) equals a constant, functions \( r \) and \( \theta \) are introduced to seek a solution such that

\[
(1) \quad y(x) = r(x) \sin(f(x) + \theta(x)); \quad y'(x) = r(x)f'(x) \cos(f(x) + \theta(x)).
\]

This requires that

\[
(2) \quad \sin(f(x) + \theta(x)) \frac{r'(x)}{r(x)} + \cos(f(x) + \theta(x))\theta'(x) = 0,
\]

while substitution into \(-y'' = (f')^2y\) yields

\[
(3) \quad \cos(f(x) + \theta(x)) \frac{r'(x)}{r(x)} - \sin(f(x) + \theta(x))\theta'(x) = -F(x) \cos(f(x) + \theta(x)) \quad (x \in J).
\]

Solving (2) and (3) for \( \theta' \) and \( r'/r \) gives

\[
(4) \quad 2\theta'(x) = F(x) \sin 2(f(x) + \theta(x)); \quad \frac{r'(x)}{r(x)} = -F(x) \cos^2(f(x) + \theta(x)) \quad (x \in J).
\]

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Since \( \int_0^\infty |F| < \infty \), the first equation in (4) has a solution, say \( \theta_1 \), such that \( \theta_1 \to 0 \) as \( x \to \infty \). Put \( r_1 \exp -\int x F \cos(\theta_1) \). Then \( r_1 \) and \( \theta_1 \) constitute a solution of (4). Set \( y_1 = r_1 \sin(\theta_1) \). One can construct \( y_2 \) similarly.

\section*{References}


\textbf{REMARK ON \( \pi(x) = o(x) \)}

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In Hardy and Wright [1], Landau [2], and Prachar [3] there are proofs of \( \pi(x) = o(x) \). The purpose of this paper is to provide a simpler proof.

\textbf{Theorem.} \( \pi(x) = o(x) \).

\textbf{Proof.} Let \( M_r \) denote the product of the first \( r \) primes. Given any positive real number \( x \geq 2 \), it is clear that there exist unique positive integers \( k \) and \( r \), with \( 1 \leq k \leq p_{r+1} - 1 \), such that

\begin{equation}
(1) \quad k M_r \leq x < (k + 1) M_r.
\end{equation}

For any \( x \) satisfying (1) we must also have

\begin{equation}
(2) \quad \pi(x) \leq (k + 1) \phi(M_r) + r
\end{equation}

(where \( \phi(M_r) \) is the totient of \( M_r \)). From (1) and (2) we obtain

\begin{equation}
(3) \quad 0 \leq \frac{\pi(x)}{x} \leq \frac{(k + 1) \phi(M_r)}{k M_r} + \frac{r}{k M_r}.
\end{equation}

Since \( k M_r \geq kr! \geq r! \) and \( (k + 1)/k \leq 2 \), we replace (3) by

\begin{equation}
(4) \quad 0 \leq \frac{\pi(x)}{x} \leq 2 \prod_{i=1}^{r} (1 - \frac{1}{p_i}) + \frac{1}{(r - 1)!}.
\end{equation}