THE NUMBER OF FULL SETS WITH n ELEMENTS

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1. Introduction. This paper gives a recursion for $F(n)$, the number of full sets with $n$ elements. Using a function discovered by Ackermann [1] we give a well-ordering for the elements of any finite full set, and then construct a bijection between the full sets of order $n$ and the vectors of order $n-1$ whose positive integral coefficients satisfy $a_1 = 1$, $a_{i-1} < a_i < 2^i$ for $i = 2, \ldots, n - 1$. These vectors are then counted.

A set $x$ is full if every element of $x$ is also a subset of $x$. The terminology follows the appendix on set theory in Kelley [2]. Throughout, $\emptyset$ denotes the null set, and 0 denotes the number zero. Without loss of generality we assume the full sets encountered in the proofs have cardinality 2 or greater.

2. The ordering function. We first prove

**Lemma 1.** If $x$ is full, $y \subseteq x$, $y \neq x$, then there is $y_1 \in x$ with $y_1 \subseteq y$ but $y_1 \notin y$.

**Proof.** Assume $y \subseteq x$ and $y \neq x$. Then $x - y \neq \emptyset$. By the axiom of regularity there is a $y_1 \in x - y$ where $y_1 \cap (x - y) = \emptyset$. Since $x$ is full, $y_1 \subseteq x$ and hence $y_1 \subseteq y$. Note that if $x$ is full and $x \neq \emptyset$, then $\emptyset \in x$. We state without proof the next

**Lemma 2.** If $x$ is a finite full set, then there is a unique $g : x \rightarrow J$ (where $J$ is the set of non-negative integers) satisfying

$$g(\emptyset) = 0, \quad \text{and for each } y \in x, \quad g(y) = \sum_{s \in y} 2^s.$$  

**Theorem 1.** $g : x \rightarrow J$ is an injection.

**Proof.** It will be sufficient to show that if $y \subseteq x$, $\emptyset \subseteq y$, $y \neq x$, and the restriction $g : y \rightarrow J$ is injective, then $g : y \cup \{y_1\} \rightarrow J$ is injective, where $y_1$ refers to the element of Lemma 1. Observe that $g(y_1) = \sum_{s \in y_1} 2^s$. Since $2^s > a$ for any real $a \geq 0$, $g(y_1) > g(z)$ for each $z \subseteq y_1$. Thus $g : y \cup \{y_1\} \rightarrow J$ is injective.

We may now order any finite full set $x$ in the form

$$x = \{x_0, x_1, \ldots, x_n\} \quad \text{by the demand} \quad g(x_i) < g(x_{i+1}),$$

$$i = 0, \ldots, n - 1.$$
Since the functions $g$ agree on the intersections of their domains, a well-ordering (of type $\omega$) is also obtained for the union of all finite full sets.

3. **Full sets and vectors.** If $x$ is a finite full set, write $x = \{x_0, \ldots, x_n\}$ using (2.2) and define $h: x \to J$ inductively by

$$h(\emptyset) = 0, \text{ and for each } x_i \in x, h(x_i) = \sum_{x_j \in x_i} 2^i.$$ 

The following lemma shows $h$ to be defined.

**Lemma 3.** $x_{k+1} \subseteq \{x_0, \ldots, x_k\}.$

**Proof.** Assume the lemma false. Then there is $x_j \in x_{k+1}$ with $j > k+1.$ Since $x_j \in x_{k+1}, g(x_j) < g(x_{k+1}).$ But by (2.2), $g(x_j) > g(x_{k+1}).$

**Theorem 2.** $h$ is injective.

**Proof.** The proof is similar to that of Theorem 1. Assume $h$ is injective when restricted to some $y \subseteq x, y = \{x_0, \ldots, x_k\}.$ When $h(x_{k+1})$ is represented as the sum of powers of two, the powers are the subscripts of elements of $y$ and hence determine the members of $x_{k+1}.$

Let $X_{n+1}$ be the class of all full sets of cardinality $n+1,$ where $n \geq 1,$ and let $V_n$ be the $n$-fold Cartesian product of $J.$ Define $H: X_{n+1} \to V_n$ by

$$H(x) = (h(x_1), \ldots, h(x_n)), \quad \text{where} \quad x = \{x_0, \ldots, x_n\}.$$ 

Clearly $H$ is defined for any full set $x \in X_{n+1},$ and it follows that

**Theorem 3.** $H$ is injective.

Now let $A_n$ be the set of vectors of order $n$ with positive integral coefficients that satisfy

$$(3.1) \quad a_1 = 1, \quad \text{and} \quad a_{i-1} < a_i < 2^i \quad \text{for } i = 2, \ldots, n.$$ 

**Theorem 4.** $H: X_{n+1} \to A_n$ is a bijection.

**Proof.** We first prove that range $H \subseteq A_n.$ Take an arbitrary $x \in X_{n+1},$ where $x = \{x_0, \ldots, x_n\}.$ Clearly $h(x_i) \in J$ for all $i \leq n.$ Since $x_1 = \{\emptyset\}, h(x_1) = a_1 = 1.$ Now if $i < j,$ then $x_j \in x_i$ by (2.1) and (2.3). Then by Lemma 3, there is $x_k \in x_j$ with $k > i$ and hence $h(x_k) > h(x_i)$ and the left inequality of (3.1) is satisfied. For an arbitrary $x_i, h(x_i)$ will be maximum if $x_i$ has for elements all of the $x_k$'s for $k < i.$ But $\sum_{k=0}^{i-1} 2^k < 2^i,$ and the right inequality of (3.1) is satisfied.

It remains only to show that $H$ is surjective. Given any $A \in A_n,$ we will construct a full set $x$ such that $H(x) = A.$ Say $A = (a_1, \ldots, a_n).$
Let $x_0 = \emptyset$ and $x_1 = \{\emptyset\}$. Suppose that the $x$s are determined for $i < k$ such that $h(x_i) = a_i$ for all $i < k$. Then let $x_k = \{x_j : j \in D(a_k)\}$ where $D(a_k)$ is the unique set of integers such that $\sum_{n \in D(a_k)} 2^n = a_k$. Then, since $a_k < 2^k$, $y \in D(a_k)$ implies that $0 \leq y < k$. Observe that this ordering of $x$ agrees with (2.2). $x_k$ is uniquely determined and $h(x_k) = a_k$. The set $x = \{x_0, \ldots, x_n\}$ is full, and $h(x_i) = a_i$ for $i = 1, \ldots, n$.

4. The counting recursion. Let $f(n, k)$ be the number of distinct vectors in $A_n$ whose $n$th coefficient is $k$. There are as many such vectors as there are vectors in $A_{n-1}$ whose $(n-1)$th coefficient is less than $k$ (with each vector in $A_{n-1}$, provided $a_{n-1} < k$, form the vector in $A_n$ by adjoining $a_n = k$). That is, $f(n, k) = \sum_{i<k} f(n-1, i)$, where $f(n, k) = 0$ if $k < n$ or if $k \geq 2^n$, and $f(1, 1) = 1$.

We then obtain the related function $F(n+1) = \sum_k f(n, k)$. The author wishes to thank John Riordan, who has reduced this related function to a recursive formula for $F(n+1)$ in terms of $F(i)$, $i = 1, \ldots, n-1$.

Consider the generating function $f_2(x) = \sum_n f(n, k)x^n$. Observe that $f_{2}+(x) = (1+x)f_{2}(x)$, $0 \leq j < 2^k$. Write $f_{2}(x) = x^{k+1}g_{k}(x)$. Then $g_{1}(x) = 1$, $g_{2}(x) = 2 + x = [(1+x)^2 - 1]x^{-1}$, and in general,

$$xg_{k+1}(x) = (1 + x)^{k+1}g_{k}(x) - g_{k}(0), \quad K = 2^k, \quad k = 2, 3, \ldots$$

By iteration (and $xg_{2}(x) = (1+x)^2 - 1$),

$$x^{n-1}g_{n}(x) = (1 + x)^{N-2} - \sum_{j=1}^{n-2} g_{j}(0)x^{j-1}(1 + x)^{N-2j+1} - x^{n-2}g_{n-1}(0), \quad N = 2^n.$$

Then

$$g_{n}(0) = \binom{N - 2}{n - 1} - \sum_{j=1}^{n-2} \frac{g_{j}(0)}{n-j} \binom{N - 2j+1}{n-1}, \quad n = 3, 4, \ldots \quad (g_{1}(0) = 1, \ g_{2}(0) = 2).$$

The constant term of $g_{n}(x)$ is $f(n+1, 2^n)$. Noting that $F(n+1) = f(n+1, 2^n)$, we obtain $g_{n}(0) = F(n+1)$, which, by substitution, gives

$$F(n+1) = \binom{2^n - 2}{n - 1} - \sum_{j=1}^{n-2} \frac{F(j + 1)}{n-j} \binom{2^n - 2j+1}{n-1}, \quad n = 3, 4, \ldots.$$

We conclude with a list of values for $F(n)$ from $n = 1$ to $n = 10$. 

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A NOTE ON THE GREATEST CROSSNORM

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Schatten has shown [5, Lemma 2, p. 323; 6, Lemma 3.7, p. 55] that, if $M$ is a closed subspace of a Banach space $B$, and there is a projection of $B$ onto $M$ with bound unity, then the greatest cross-norm on the tensor product $B \odot M$ is an extension of the greatest crossnorm on $M \odot M$ for any Banach space $M$.

Now it is known that there is a projection with bound unity of the second conjugate $B^{**}$ of a Banach space $B$ onto $B_0$ (the canonical image of $B$ in $B^{**}$) for conjugate spaces $B$ and for some others [3, p. 580], though not for all Banach spaces (cf. [7]). For such spaces, then, the greatest crossnorm on $B^{**} \odot M$ is an extension of the greatest crossnorm on $B_0 \odot M$. The purpose of this note is to show that the restriction to such spaces is unnecessary. (N.B. $B$ is sometimes embedded in $B^{**}$ by identifying it with $B_0$.)

**Theorem.** Let $B$ and $M$ be any Banach spaces. Then the greatest crossnorm on $B^{**} \odot M$ is an extension of the greatest crossnorm on $B_0 \odot M$ (where $B_0$ is the canonical image of $B$ in $B^{**}$).

Let $x$ be any element of $B_0 \odot M \subset B^{**} \odot M$. Clearly (in the notation of [2, §2.4, pp. 347–351])

$$\gamma(B^{**} \odot M)(x) \leq \gamma(B_0 \odot M)(x)$$

(since the infimum on the left-hand side is taken over a larger collection of expressions). On the other hand, there exists a continuous

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