THE NUMBER OF FULL SETS WITH n ELEMENTS

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1. **Introduction.** This paper gives a recursion for $F(n)$, the number of full sets with n elements. Using a function discovered by Ackermann [1] we give a well-ordering for the elements of any finite full set, and then construct a bijection between the full sets of order n and the vectors of order $n-1$ whose positive integral coefficients satisfy $a_1=1$, $a_{i-1}<a_i<2^i$ for $i=2, \cdots, n-1$. These vectors are then counted.

A set x is full if every element of x is also a subset of x. The terminology follows the appendix on set theory in Kelley [2]. Throughout, $\emptyset$ denotes the null set, and 0 denotes the number zero. Without loss of generality we assume the full sets encountered in the proofs have cardinality 2 or greater.

2. **The ordering function.** We first prove

LEMMA 1. *If x is full, $y\subset x$, $y\neq x$, then there is $y_1\in x$ with $y_1\subset y$ but $y_1\notin y$.*

PROOF. Assume $y\subset x$ and $y\neq x$. Then $x-y\neq\emptyset$. By the axiom of regularity there is a $y_1\in x-y$ where $y_1\cap(x-y)=\emptyset$. Since x is full, $y_1\subset x$ and hence $y_1\subset y$. Note that if x is full and $x\neq\emptyset$, then $\emptyset\in x$. We state without proof the next

LEMMA 2. *If x is a finite full set, then there is a unique $g: x\rightarrow J$ (where J is the set of non-negative integers) satisfying*

$$(2.1)\qquad g(\emptyset) = 0, \quad \textit{and for each } y \in x, \quad g(y) = \sum_{z\in y} 2^{g(z)}.$$

THEOREM 1. *$g: x\rightarrow J$ is an injection.*

PROOF. It will be sufficient to show that if $y\subset x$, $\emptyset\in y$, $y\neq x$, and the restriction $g: y\rightarrow J$ is injective, then $g: y\cup\{y_1\}\rightarrow J$ is injective, where y_1 refers to the element of Lemma 1. Observe that $g(y_1)=\sum_{z\in y} 2^{g(z)}$. Since $2^a>a$ for any real $a\geqq 0$, $g(y_1)>g(z)$ for each $z\in y_1$. Thus $g: y\cup\{y_1\}\rightarrow J$ is injective.

We may now order any finite full set x in the form

$$(2.2)\qquad x = \{x_0, x_1, \cdots, x_n\} \quad \text{by the demand} \quad g(x_i) < g(x_{i+1}), \quad i = 0, \cdots, n-1.$$

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Since the functions g agree on the intersections of their domains, a well-ordering (of type ω) is also obtained for the union of all finite full sets.

3. **Full sets and vectors.** If x is a finite full set, write $x = \{x_0, \cdots, x_n\}$ using (2.2) and define $h: x \to J$ inductively by

$$h(\varnothing) = 0, \text{ and for each } x_i \in x, \; h(x_i) = \sum_{x_j \in x_i} 2^j.$$

The following lemma shows h to be defined.

LEMMA 3. $x_{k+1} \subset \{x_0, \cdots, x_k\}$.

PROOF. Assume the lemma false. Then there is $x_j \in x_{k+1}$ with $j > k+1$. Since $x_j \in x_{k+1}$, $g(x_j) < g(x_{k+1})$. But by (2.2), $g(x_j) > g(x_{k+1})$.

THEOREM 2. *h is injective.*

PROOF. The proof is similar to that of Theorem 1. Assume h is injective when restricted to some $y \subset x$, $y = \{x_0, \cdots, x_k\}$. When $h(x_{k+1})$ is represented as the sum of powers of two, the powers are the subscripts of elements of y and hence determine the members of x_{k+1}.

Let X_{n+1} be the class of all full sets of cardinality $n+1$, where $n \geqq 1$, and let V_n be the n-fold Cartesian product of J. Define $H: X_{n+1} \to V_n$ by

$$H(x) = (h(x_1), \cdots, h(x_n)), \quad \text{where} \quad x = \{x_0, \cdots, x_n\}.$$

Clearly H is defined for any full set $x \in X_{n+1}$, and it follows that

THEOREM 3. *H is injective.*

Now let A_n be the set of vectors of order n with positive integral coefficients that satisfy

$$(3.1) \qquad a_1 = 1, \quad \text{and} \quad a_{i-1} < a_i < 2^i \quad \text{for } i = 2, \cdots, n.$$

THEOREM 4. *$H: X_{n+1} \to A_n$ is a bijection.*

PROOF. We first prove that range $H \subset A_n$. Take an arbitrary $x \in X_{n+1}$, where $x = \{x_0, \cdots, x_n\}$. Clearly $h(x_i) \in J$ for all $i \leqq n$. Since $x_1 = \{\varnothing\}$, $h(x_1) = a_1 = 1$. Now if $i < j$, then $x_j \not\subset x_i$ by (2.1) and (2.3). Then by Lemma 3, there is $x_k \in x_j$ with $k > i$ and hence $h(x_j) > h(x_i)$ and the left inequality of (3.1) is satisfied. For an arbitrary x_i, $h(x_i)$ will be maximum if x_i has for elements all of the x_k's for $k < i$. But $\sum_{k=0}^{i-1} 2^k < 2^i$, and the right inequality of (3.1) is satisfied.

It remains only to show that H is surjective. Given any $A \in A_n$, we will construct a full set x such that $H(x) = A$. Say $A = (a_1, \cdots, a_n)$.

Let $x_0=\varnothing$ and $x_1=\{\varnothing\}$. Suppose that the x_is are determined for $i<k$ such that $h(x_i)=a_i$ for all $i<k$. Then let $x_k=\{x_j: j\in D(a_k)\}$ where $D(a_k)$ is the unique set of integers such that $\sum_{y\in D(a_k)} 2^y=a_k$. Then, since $a_k<2^k$, $y\in D(a_k)$ implies that $0\leqq y<k$. Observe that this ordering of x agrees with (2.2). x_k is uniquely determined and $h(x_k)=a_k$. The set $x=\{x_0, \cdots, x_n\}$ is full, and $h(x_i)=a_i$ for $i=1, \cdots, n$.

4. **The counting recursion.** Let $f(n, k)$ be the number of distinct vectors in A_n whose nth coefficient is k. There are as many such vectors as there are vectors in A_{n-1} whose $(n-1)$th coefficient is less than k (with each vector in A_{n-1}, provided $a_{n-1}<k$, form the vector in A_n by adjoining $a_n=k$). That is, $f(n, k)=\sum_{i<k} f(n-1, i)$, where $f(n, k)=0$ if $k<n$ or if $k\geqq 2^n$, and $f(1, 1)=1$.

We then obtain the related function $F(n+1)=\sum_k f(n, k)$. The author wishes to thank John Riordan, who has reduced this related function to a recursive formula for $F(n+1)$ in terms of $F(i)$, $i=1, \cdots, n-1$.

Consider the generating function $f_k(x)=\sum_n f(n, k)x^n$. Observe that $f_{2^k+j}(x)=(1+x)^j f_{2^k}(x)$, $0\leqq j<2^k$. Write $f_{2^k}(x)=x^{k+1}g_k(x)$. Then $g_1(x)=1$, $g_2(x)=2+x=[(1+x)^2-1]x^{-1}$, and in general,

$$xg_{k+1}(x) = (1+x)^K g_k(x) - g_k(0), \qquad K = 2^k,\ k = 2, 3, \cdots.$$

By iteration (and $xg_2(x)=(1+x)^2-1$),

$$x^{n-1}g_n(x) = (1+x)^{N-2} - \sum_{j=1}^{n-2} g_j(0)x^{j-1}(1+x)^{N-2^{j+1}} - x^{n-2}g_{n-1}(0), \qquad N = 2^n.$$

Then

$$g_n(0) = \binom{N-2}{n-1} - \sum_{j=1}^{n-2} g_j(0)\binom{N-2^{j+1}}{n-j}, \qquad n = 3, 4, \cdots \ (g_1(0) = 1,\ g_2(0) = 2).$$

The constant term of $g_n(x)$ is $f(n+1, 2^n)$. Noting that $F(n+1)=f(n+1, 2^n)$, we obtain $g_n(0)=F(n+1)$, which, by substitution, gives

$$F(n+1) = \binom{2^n-2}{n-1} - \sum_{j=1}^{n-2} F(j+1)\binom{2^n-2^{j+1}}{n-j}, \quad n = 3, 4, \cdots.$$

We conclude with a list of values for $F(n)$ from $n=1$ to $n=10$.

$F(1) = 1$	$F(5) = 88$	$F(9) = 1{,}097{,}780{,}312$
$F(2) = 1$	$F(6) = 1{,}802$	$F(10) = 376{,}516{,}036{,}188$
$F(3) = 2$	$F(7) = 75{,}598$	
$F(4) = 9$	$F(8) = 6{,}421{,}599$	

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2. J. L. Kelley, *General topology*, Van Nostrand, New, York, 1955.

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A NOTE ON THE GREATEST CROSSNORM

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Schatten has shown [**5**, Lemma 2, p. 323; **6**, Lemma 3.7, p. 55] that, if $\mathfrak{M}$ is a closed subspace of a Banach space $\mathfrak{B}$, and there is a projection of $\mathfrak{B}$ onto $\mathfrak{M}$ with bound unity, then the greatest crossnorm on the tensor product $\mathfrak{B} \odot \mathfrak{N}$ is an extension of the greatest crossnorm on $\mathfrak{M} \odot \mathfrak{N}$ for any Banach space $\mathfrak{N}$.

Now it is known that there is a projection with bound unity of the second conjugate $\mathfrak{B}^{**}$ of a Banach space $\mathfrak{B}$ onto $\mathfrak{B}_0$ (the canonical image of $\mathfrak{B}$ in $\mathfrak{B}^{**}$) for conjugate spaces $\mathfrak{B}$ and for some others [**3**, p. 580], though not for *all* Banach spaces (cf. [**7**]). For such spaces, then, the greatest crossnorm on $\mathfrak{B}^{**} \odot \mathfrak{N}$ is an extension of the greatest crossnorm on $\mathfrak{B}_0 \odot \mathfrak{N}$. The purpose of this note is to show that the restriction to such spaces is unnecessary. (N.B. $\mathfrak{B}$ is sometimes embedded in $\mathfrak{B}^{**}$ by identifying it with $\mathfrak{B}_0$.)

THEOREM. *Let $\mathfrak{B}$ and $\mathfrak{N}$ be any Banach spaces. Then the greatest crossnorm on $\mathfrak{B}^{**} \odot \mathfrak{N}$ is an extension of the greatest crossnorm on $\mathfrak{B}_0 \odot \mathfrak{N}$ (where $\mathfrak{B}_0$ is the canonical image of $\mathfrak{B}$ in $\mathfrak{B}^{**}$).*

Let $\mathfrak{X}$ be any element of $\mathfrak{B}_0 \odot \mathfrak{N} \subset \mathfrak{B}^{**} \odot \mathfrak{N}$. Clearly (in the notation of [**2**, §2.4, pp. 347–351])

$$\gamma\{\mathfrak{B}^{**} \odot \mathfrak{N}\}(\mathfrak{X}) \leqq \gamma\{\mathfrak{B}_0 \odot \mathfrak{N}\}(\mathfrak{X})$$

(since the infimum on the left-hand side is taken over a larger collection of expressions). On the other hand, there exists a continuous

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