ON THE COMMUTATOR SUBGROUP OF THE ORTHOGONAL GROUP OVER THE 2-ADIC NUMBERS

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1. **Introduction.** Let V be a vector space of dimension n over some field k of characteristic $\neq 2$ with an orthogonal geometry as in [1, Chapter III]. Let $O(V)$ be the orthogonal group of V, $O'(V)$ the subgroup of elements of determinant 1 and spinor-norm 1 and $\Omega(V)$ the commutator subgroup of $O(V)$. It is well known that $O'(V) = \Omega(V)$ if (i) $n \leqq 3$ or (ii) V is isotropic. If $n > 3$ and V is anisotropic this is no longer in general true. Our interest focuses on the case where k is a local field (i.e., a field complete with respect to a discrete non-archimedean valuation with finite residue class field $\bar{k}$). Then it is well known that V is isotropic if $n \geqq 5$. Hence we are left with the consideration of $n=4$ and V anisotropic. In [4] Kneser states that in this case $O'(V) \neq \Omega(V)$ and indeed it is not hard to show that $(O'(V):\Omega(V)) = 2$ if the characteristic of $\bar{k} > 2$. It is the purpose of this note to prove that $O'(V) = \Omega(V)$ when k is the field of 2-adic numbers.[1]

2. **Preliminaries.** Let us denote the symmetry with respect to the hyperplane perpendicular to the nonisotropic vector A by τ_A. We have

PROPOSITION 1. *Let V have dimension n and suppose $\sigma = \tau_{A_1}\tau_{A_2} \cdots \tau_{A_n} \in O(V)$. Define a new space $V_\sigma = \langle A_1' \rangle \perp \langle A_2' \rangle \perp \cdots \perp \langle A_n' \rangle$ by setting $(A_i')^2 = A_i^2$ for $i = 1, 2, \cdots, n$. Suppose V and V_σ are isometric. Then $\sigma \in \Omega(V) \Leftrightarrow -1_{V_\sigma} \in \Omega(V_\sigma)$.*

PROOF. Let ϕ: $V_\sigma \rightarrow V$ be an isometry. Then $\phi O(V_\sigma)\phi^{-1} = O(V)$. Set $B_i = \phi A_i'$ for $i = 1, 2, \cdots, n$. Then $\phi(-1_{V_\sigma})\phi^{-1} = \phi(\tau_{A_1'}\tau_{A_2'} \cdots \tau_{A_n'})\phi^{-1} = \tau_{\phi A_1'}\tau_{\phi A_2'} \cdots \tau_{\phi A_n'} = \tau_{B_1}\tau_{B_2} \cdots \tau_{B_n} \equiv \tau_{A_1}\tau_{A_2} \cdots \tau_{A_n} \bmod \Omega(V)$. Since $\sigma = \tau_{A_1}\tau_{A_2} \cdots \tau_{A_n}$ we are through.

PROPOSITION 2. *Let V be 4-dimensional and suppose $\sigma \in O'(V)$. Then a necessary condition that $\sigma \notin \Omega(V)$ is that σ have the form $\tau_{A_1}\tau_{A_2}\tau_{A_3}\tau_{A_4}$ and $A_1^2, A_2^2, A_3^2, A_4^2$ lie in distinct classes of k^* modulo k^{*2}.*

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[1] O. T. O'Meara informs me that he has proved this for any local field with characteristic $\bar{k} = 2$ using different methods.

PROOF. If σ is the product of two symmetries, $\sigma=\tau_{A_1}\tau_{A_2}$, then $\sigma\in\Omega(V)$ as is shown in [1, Theorem 5.14]. Hence σ has the form $\tau_{A_1}\tau_{A_2}\tau_{A_3}\tau_{A_4}$. If the A_i^2 do not lie in distinct classes of k^* modulo k^{*2} we may assume (since the issue is mod $\Omega(V)$) that $A_1^2=A_2^2$. Then by Witt's Theorem, there exists $\lambda\in O(V)$ such that $A_2=\lambda A_1$. Thus $\sigma=\tau_{A_1}\tau_{A_2}\tau_{A_3}\tau_{A_4}=\tau_{A_1}\tau_{\lambda A_1}\tau_{A_3}\tau_{A_4}=\tau_{A_1}\lambda\tau_{A_1}\lambda^{-1}\tau_{A_3}\tau_{A_4}\equiv\tau_{A_3}\tau_{A_4} \mod \Omega(V)$ and we are in the situation already dealt with at the beginning of the proof.

Although our main concern is over the field of 2-adic numbers we now prove, for the sake of completeness, the following

THEOREM 1. *Let k be a local field with residue class field $\bar{k}$ of characteristic >2. Let V be a 4-dimensional anisotropic space over k. Then* $(O'(V):\Omega(V))=2$.

PROOF. Since $(k^*:k^{*2})=4$, it follows immediately from Proposition 2 that $(O'(V):\Omega(V))\leqq 2$. We may choose as representatives for k^* modulo k^{*2} $1, \nu, \pi, \nu\pi$ where ν is a nonsquare unit and π is a prime. Since V is anisotropic, we may write V in the form $V=\langle A_1\rangle\perp\langle A_2\rangle \perp \langle A_3\rangle \perp \langle A_4\rangle$ with $A_1^2=1$, $A_2^2=-\nu$, $A_3^2=\pi$ and $A_4^2=-\pi\nu$. Let $U=\langle A_1\rangle\perp\langle A_2\rangle$. There exists $B\in U$ such that $B^2=\nu$ as one easily verifies. Likewise, there exists $C\in U^*$ with $C^2=\pi\nu$. Set $\sigma=\tau_{A_1}\tau_B\tau_{A_3}\tau_C$. Then Dieudonne's technique (see [2, p. 93]) suitably modified shows that $\sigma\notin\Omega(V)$ and the theorem is proved.

3. **Main result.** We now assume that V is a 4-dimensional vector space over the field of 2-adic numbers. Furthermore we assume that V possesses an orthogonal geometry that is anisotropic and note for future use that V is unique up to isometry. (For a proof, see [3, Satz 7.3].)

LEMMA. $-1_V\in\Omega(V)$.

PROOF. We may set $V=\langle A_1\rangle\perp\langle A_2\rangle\perp\langle A_3\rangle\perp\langle A_4\rangle$ with $A_i^2=1$ for $i=1, 2, 3, 4$. Then $-1_V=\tau_{A_1}\tau_{A_2}\tau_{A_3}\tau_{A_4}$ and hence is in $\Omega(V)$ by Proposition 2.

THEOREM 2. $O'(V)=\Omega(V)$.

PROOF. It suffices to show that $O'(V)\subseteq\Omega(V)$. Thus let $\sigma\in O'(V)$. By Proposition 2 we may assume $\sigma=\tau_{A_1}\tau_{A_2}\tau_{A_3}\tau_{A_4}$ with the A_i^2 in different classes of k^* modulo k^{*2}. We may take 1, 3, 5, 7, 2, 6, 10, 14 as representatives of k^* modulo k^{*2} and note that there are exactly 14 possibilities σ_i for σ defined by the set $\{A_1^2, A_2^2, A_3^2, A_4^2\}$. We list these and also note whether or not the corresponding V_{σ_i} of Proposition 1 is anisotropic by an asterisk.

i of σ_i	$\{A_1^2, A_2^2, A_3^2, A_4^2\}$
1	1, 3, 5, 7
2	2, 6, 10, 14
3*	1, 3, 2, 6
4*	1, 3, 10, 14
5	1, 5, 2, 10
6*	1, 5, 6, 14
7	1, 7, 2, 14
8	1, 7, 6, 10
9	3, 5, 6, 10
10	3, 5, 2, 14
11	3, 7, 6, 14
12*	3, 7, 2, 10
13*	5, 7, 10, 14
14*	5, 7, 2, 6 .

By Proposition 1, the lemma and the fact, already noted, that all 4-dimensional anisotropic spaces are isometric we see that σ_3, σ_4, σ_6, σ_{12}, σ_{13} and $\sigma_{14} \in \Omega(V)$. To complete the proof first note that $\sigma \equiv \tau \bmod \Omega(V) \Leftrightarrow \sigma\tau \in \Omega(V)$. Now $\sigma_1\sigma_3 \equiv \sigma_{14} \bmod \Omega(V)$. But σ_3 and σ_{14} are in $\Omega(V)$. Hence, $\sigma_1 \in \Omega(V)$. Similar computations show that the remaining σ_i lie in $\Omega(V)$ and the theorem is proved.

References

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