ON THE COMMUTATOR SUBGROUP OF THE ORTHOGONAL GROUP OVER THE 2-ADIC NUMBERS

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1. Introduction. Let $V$ be a vector space of dimension $n$ over some field $k$ of characteristic $\neq 2$ with an orthogonal geometry as in [1, Chapter III]. Let $O(V)$ be the orthogonal group of $V$, $O'(V)$ the subgroup of elements of determinant 1 and spinor-norm 1 and $\Omega(V)$ the commutator subgroup of $O(V)$. It is well known that $O'(V) = \Omega(V)$ if (i) $n \leq 3$ or (ii) $V$ is isotropic. If $n > 3$ and $V$ is anisotropic this is no longer in general true. Our interest focuses on the case where $k$ is a local field (i.e., a field complete with respect to a discrete non-archimedean valuation with finite residue class field $\mathfrak{k}$). Then it is well known that $V$ is isotropic if $n \geq 5$. Hence we are left with the consideration of $n = 4$ and $V$ anisotropic. In [4] Kneser states that in this case $O'(V) \nsubseteq \Omega(V)$ and indeed it is not hard to show that $(O'(V) : \Omega(V)) = 2$ if the characteristic of $\mathfrak{k} > 2$. It is the purpose of this note to prove that $O'(V) = \Omega(V)$ when $k$ is the field of 2-adic numbers.1

2. Preliminaries. Let us denote the symmetry with respect to the hyperplane perpendicular to the nonisotropic vector $A$ by $\tau_A$. We have

**Proposition 1.** Let $V$ have dimension $n$ and suppose $\sigma = \tau_{A_1}\tau_{A_2}\cdots \tau_{A_n} \in O(V)$. Define a new space $V_o = \langle A_1 \rangle \perp \langle A_2 \rangle \perp \cdots \perp \langle A_n \rangle$ by setting $(A_i)^2 = A_i^2$ for $i = 1, 2, \cdots, n$. Suppose $V$ and $V_o$ are isometric. Then $\sigma \in \Omega(V) \iff -1_{V_o} \in \Omega(V_o)$.

**Proof.** Let $\phi: V_o \to V$ be an isometry. Then $\phi O(V_o)\phi^{-1} = O(V)$. Set $B_i = \phi A_i$ for $i = 1, 2, \cdots, n$. Then $\phi(-1_{V_o})\phi^{-1} = \phi(\tau_{A_1}\tau_{A_2}\cdots \tau_{A_n})\phi^{-1} = \tau_{A_1}\tau_{A_2}\cdots \tau_{A_n} \equiv \tau_{A_1}\tau_{A_2}\cdots \tau_{A_n} \mod \Omega(V)$. Since $\sigma = \tau_{A_1}\tau_{A_2}\cdots \tau_{A_n}$ we are through.

**Proposition 2.** Let $V$ be 4-dimensional and suppose $\sigma \in O'(V)$. Then a necessary condition that $\sigma \in \Omega(V)$ is that $\sigma$ have the form $\tau_{A_1}\tau_{A_2}\tau_{A_3}\tau_{A_4}$ and $A_1^2, A_2^2, A_3^2, A_4^2$ lie in distinct classes of $k^* \mod k^{*2}$.

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1 O. T. O'Meara informs me that he has proved this for any local field with characteristic $k = 2$ using different methods.
Proof. If $\sigma$ is the product of two symmetries, $\sigma = \tau_A \tau_A^t$, then $\sigma \in \Omega(V)$ as is shown in [1, Theorem 5.14]. Hence $\sigma$ has the form $\tau_A \tau_A \tau_A \tau_A$. If the $A_i$ do not lie in distinct classes of $k^*$ modulo $k^{*2}$ we may assume (since the issue is mod $\Omega(V)$) that $A_i^2 = A_i^2$. Then by Witt's theorem, there exists $\lambda \in O(V)$ such that $A_2 = \lambda A_1$. Thus $\sigma = \tau_A \tau_A \tau_A \tau_A \tau_A = \tau_A \lambda \tau_A \lambda^{-1} \tau_A \tau_A \tau_A \tau_A$ mod $\Omega(V)$ and we are in the situation already dealt with at the beginning of the proof.

Although our main concern is over the field of 2-adic numbers we now prove, for the sake of completeness, the following

**Theorem 1.** Let $k$ be a local field with residue class field $\bar{k}$ of characteristic $>2$. Let $V$ be a 4-dimensional anisotropic space over $k$. Then $(O'(V): \Omega(V)) = 2$.

**Proof.** Since $\mathbb{k}^* : k^*2) = 4$, it follows immediately from Proposition 2 that $(O'(V): \Omega(V)) \leq 2$. We may choose as representatives for $k^*$ modulo $k^{*2}$ 1, $\nu$, $\pi$, $\nu \pi$ where $\nu$ is a nonsquare unit and $\pi$ is a prime. Since $V$ is anisotropic, we may write $V$ in the form $V = \langle A_i \rangle \perp \langle A_j \rangle \perp \langle A_k \rangle \perp \langle A_l \rangle$ where $A_i^2 = 1$, $A_j^2 = -\nu$, $A_k^2 = \pi$ and $A_l^2 = -\pi \nu$. Let $U = \langle A_1 \rangle \perp \langle A_3 \rangle$. There exists $B \in U$ such that $B^2 = \nu$ as one easily verifies. Likewise, there exists $C \in U^*$ with $C^2 = \pi \nu$. Set $\sigma = \tau_A \tau_B \tau_{A_3} \tau_{A_4}$. Then Dieudonné's technique (see [2, p. 93]) suitably modified shows that $\sigma \in \Omega(V)$ and the theorem is proved.

3. **Main result.** We now assume that $V$ is a 4-dimensional vector space over the field of 2-adic numbers. Furthermore we assume that $V$ possesses an orthogonal geometry that is anisotropic and note for future use that $V$ is unique up to isometry. (For a proof, see [3, Satz 7.3].)

**Lemma.** $-1 \in \Omega(V)$.

**Proof.** We may set $V = \langle A_1 \rangle \perp \langle A_2 \rangle \perp \langle A_3 \rangle \perp \langle A_4 \rangle$ with $A_i^2 = 1$ for $i = 1, 2, 3, 4$. Then $-1 = \tau_A \tau_A \tau_A \tau_A$ and hence is in $\Omega(V)$ by Proposition 2.

**Theorem 2.** $O'(V) = \Omega(V)$.

**Proof.** It suffices to show that $O'(V) \subseteq \Omega(V)$. Thus let $\sigma \in O'(V)$. By Proposition 2 we may assume $\sigma = \tau_A \tau_A \tau_A \tau_A$ with the $A_i^2$ in different classes of $k^*$ modulo $k^{*2}$. We may take 1, 3, 5, 7, 2, 6, 10, 14 as representatives of $k^*$ modulo $k^{*2}$ and note that there are exactly 14 possibilities $\sigma_i$ for $\sigma$ defined by the set $\{A_1^2, A_2^2, A_3^2, A_4^2\}$. We list these and also note whether or not the corresponding $V_{\sigma_i}$ of Proposition 1 is anisotropic by an asterisk.
By Proposition 1, the lemma and the fact, already noted, that all 4-dimensional anisotropic spaces are isometric we see that $\sigma_3, \sigma_4, \sigma_6, \sigma_{13}, \sigma_{13}$ and $\sigma_{14} \in \Omega(V)$. To complete the proof first note that $\sigma \equiv \tau \mod \Omega(V) \iff \sigma \tau \in \Omega(V)$. Now $\sigma_3 \sigma_3 \equiv \sigma_{14} \mod \Omega(V)$. But $\sigma_3$ and $\sigma_{14}$ are in $\Omega(V)$. Hence, $\sigma_3 \in \Omega(V)$. Similar computations show that the remaining $\sigma_i$ lie in $\Omega(V)$ and the theorem is proved.

### References


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