A CHARACTERIZATION OF CERTAIN QUASI-OPEN MAPPINGS

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1. **Introduction.** In this paper we shall deal with mappings (continuous functions) defined on regions (open and connected) subsets of a plane P with the range of the functions also contained in P. In reference [1], M. K. Fort defined such a mapping f to be *minimal* provided that for each closed 2-cell N contained in the domain of f, $f(N) \subset g(N)$ for every mapping g whose domain contains N and which is such that $f|\operatorname{Fr} N = g|\operatorname{Fr} N$. The following result was obtained in [1].

If f is light, then f is open if and only if f is minimal.

In this paper, we prove the following similar theorem for certain quasi-open mappings.

1.1. THEOREM. *Let $f: X \to P$ be a compact mapping defined on a simply connected region X in a plane P with $f(X) \subset P$. Then f is quasi-open if and only if f is minimal and for no $x \in f(X)$ is it true that $f^{-1}(x)$ separates X.*

The sufficiency of the condition in 1.1 is established in §2 and the necessity in §3.

Recall that a mapping $f: X \to P$ is compact provided that for each compact set $K \subset f(X)$, $f^{-1}(K)$ is compact. f is quasi-open provided that for any $y \in f(X)$ and any open set U in X containing a compact component of $f^{-1}(y)$, $y \in \operatorname{int} f(U)$. Here and throughout the paper such terms as *interior*, *closed*, *closure* (cl), *boundary* (Fr) will always be relative to the containing plane.

With $f: X \to P$ a mapping into a plane P, if R is an *elementary region* (a bounded plane region whose boundary consists of a finite number of simple closed curves) whose closure is in the domain of f and $x \in P - f(\operatorname{Fr} R)$, then there is defined the topological index or winding number $\mu(x, f, R)$. Use will be made of the following well-known properties of the index.

1.2. *$\mu(x, f, R)$ is constant on every component of $P - f(\operatorname{Fr} R)$.*

1.3. *Suppose $x \in P - f(\operatorname{Fr} R)$. If $g: X \to P$ is such that $g(\bar{x}) = f(\bar{x})$ for $\bar{x} \in \operatorname{Fr}(R)$, then $\mu(x, f, R) = \mu(x, g, R)$.*

1.4. *Let R_j; $j = 1, 2, \cdots, k$ be a sequence of pair-wise disjoint ele-*

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mentary regions in R such that

$$(\mathrm{cl}\, R)\cdot f^{-1}(x) \subset \bigcup_j R_j.$$

Then $\mu(x, f, R) = \sum_j \mu(x, f, R_j)$. (See Theorem 3 on p. 126 in [7].)

1.5. *If* $\mu(x, f, R) \neq 0$, *then* $x \in f(R)$.

1.6. *If N is a closed 2-cell and f is a homeomorphism on N, then* $\mu(x, f, \mathrm{int}\, N) = \pm 1$ *for each* $x \in f(\mathrm{int}\, N)$.

In terms of this index, Fort [1] proved the following.

1.7. *With $f: X \to P$ as before, f is minimal if and only if for each closed 2-cell N,* $f(N) = f(\mathrm{Fr}\, N) + \{p \mid \mu(p, f, \mathrm{int}\, N) \neq 0\}$.

2. **Compact plane mappings.** In this section we make the following assumption.

2.1. *$f: X \to P$ is a compact mapping from a simply connected region in a plane P into P.*

From a result of Whyburn (§10 in [4]), there exists a factorization $f = lm$ in which m is monotone and compact and l is light and compact. Throughout the middle space $m(X)$ will be designated as M.

If the further assumption is made that for no $x \in f(X)$ it is true that $f^{-1}(x)$ separates X, it can be shown that no component of $f^{-1}(x)$ separates X. But then for no $z \in M$, is it true that $m^{-1}(z)$ separates X. 2.3 below then follows from the following result of Whyburn. (See Note 1, p. 313 in [5].)

If m is a monotone mapping from a plane X onto a space Y, then Y is a plane if and only if m is compact and $m^{-1}(y)$ does not separate X for each $y \in Y$.

2.3. *Let f be as in 2.1 and let $f = lm$ be a factorization as before. In addition assume that for $x \in f(X)$, $f^{-1}(x)$ does not separate X. Then the middle space M is a topological plane.*

2.4. *Let $f: X \to P$, l, m, M be as in 2.3. Then if N is a closed 2-cell such that $N \subset M$, $m^{-1}(\mathrm{int}\, N)$ is a simply connected region.*

PROOF. $M - \mathrm{int}\, N$ is connected. Therefore, since m is compact and monotone, $m^{-1}(M - \mathrm{int}\, N) = X - m^{-1}(\mathrm{int}\, N)$ is connected. (See 8.3 in [4].) Since m is compact, $m^{-1}(N)$ and thus cl $(m^{-1}(\mathrm{int}\, N))$ is compact. Thus $m^{-1}(\mathrm{int}\, N)$ is a plane region whose closure is compact and whose complement in the plane is connected. Hence $m^{-1}(\mathrm{int}\, N)$ is simply connected.

2.5. *Let $f: X \to P$ be a compact mapping. Let $x \in f(X)$ and let U be an open set such that $f^{-1}(x) \subset U$. Then there exists an open set $V \subset P$ such that $x \in V$ and $f^{-1}(V) \subset U$.*

This follows easily from the compactness of f.

2.6. *Let $f: X \to P$ be a mapping as in 2.3. Furthermore assume that f*

is minimal. Then f is quasi-open.

PROOF. Let $f=lm$ be a factorization as in 2.3. By 2.3 the middle space M may be taken to be a complex plane. Let $x\in f(X)$ and suppose K is a component of $f^{-1}(x)$. Let U be an open set in X such that $K\subset U$. From the definition of quasi-open, the proof will be complete if we show that x is an interior point of $f(U)$. Toward this end, let $p=m(K)$. Note that $p\in l^{-1}(x)$. Since $l^{-1}(x)$ is totally disconnected it can be shown that there exists arbitrarily small 2-cells N such that $(\mathrm{Fr}\, N)\cdot l^{-1}(x)=\emptyset$ and $p\in\mathrm{int}\, N$. By 2.5, N may be chosen so that $m^{-1}(N)\subset U$. Let $G\equiv m^{-1}(\mathrm{int}\, N)$. By 2.4, G is simply connected. Since f is compact, $f^{-1}(x)\cdot\mathrm{cl}\, G$ is compact and since $f^{-1}(x)\cdot\mathrm{Fr}\, G=\emptyset$ it follows that $f^{-1}(x)\cdot G$ is compact. Then since G is simply connected, there exists a closed 2-cell T such that $f^{-1}(x)\cdot G\subset\mathrm{int}\, T\subset T\subset G$. Further by 1.7 $\mu(x, f, \mathrm{int}\, T)\neq 0$. But then from 1.2 and 1.5, $f(T)\supset Q_x$ where Q_x is the component of $P-f(\mathrm{Fr}\, T)$ that contains x. Since Q_x is open and $x\in Q_x\subset f(T)\subset f(U)$ it follows that x is an interior point of $f(U)$.

3. **Compact quasi-open mappings.** In this section we make the following assumption.

3.1. *$f: X\to P$ is a compact and quasi-open mapping defined on a simply connected region X in the plane P and $f(X)\subset P$.*

Assuming 3.1, it is known (10.2 and 10.4 in [**4**]) that there is a monotone-light open factorization $f=lm$ for which m is a compact monotone mapping and l is compact, light, and open relative to $f(X)$. Further from the definition of quasi-open as given in the introduction, $f(X)$ is open in P and hence l is open relative to P. Furthermore, we have the following.

3.2. *For no point $x\in f(X)$ is it true that $f^{-1}(x)$ separates X.*

PROOF. Suppose $f^{-1}(x)$ separates X. Then there exists a separation

$$X - f^{-1}(x) = A + B.$$

Since $f^{-1}(x)$ is compact $A+f^{-1}(x)$ or $B+f^{-1}(x)$ is compact. Assume that $A+f^{-1}(x)$ is compact. Then $f(A)+x$ is compact. However A is open and is the union of components of point inverses. Hence from the quasi-openness of f, $f(A)$ is open. This is a contradiction since then x would be the boundary of an open set $f(A)$ in the plane.

On the basis of 3.2 *and* 2.3 *with f as in* 3.1, *the middle space M can be taken to be a plane.* Recall that $f(X)$ is open in P. Thus there is available the following result of Whyburn.

3.3. (See VIII, 1.1 and 1.11 in [**3**].) *There exists an integer k and a completely scattered set $D\subset f(X)$, such that for each $x\in f(X)-D$, $l^{-1}(x)$ consists of k distinct points. Furthermore on the set $M-l^{-1}(D)$, l is a*

local homeomorphism. [Note that for each point $x \in D$, x is an isolated point of D. Also D is closed.]

By making use of 3.3 and 2.4, the following can be obtained.

3.4. *For each $z \in f(X) - D$, there exists arbitrarily small closed 2-cells N such that $z \in \operatorname{int} N \subset N \subset f(X) - D$ and such that $f^{-1}(\operatorname{int} N)$ is the union of a finite number of pairwise disjoint open two-cells Q_i for which $f(Q_i) = \operatorname{int} N$ and $f^{-1}(z^*) \cdot Q_i$ consists of a single component of $f^{-1}(z^*)$ for each $z^* \in \operatorname{int} N$.*

3.5. *$X - f^{-1}(D)$ is a connected open set.*

PROOF. Notice that $X - f^{-1}(D) = m^{-1}(M - l^{-1}(D))$ is open and furthermore is connected since $M - l^{-1}(D)$ is connected and m is a compact monotone mapping.

3.6. DEFINITION OF $\mu(x)$. On the open connected set $S \equiv X - f^{-1}(D)$ define the function μ as follows. Let $x \in S$. Then $f^{-1}f(x)$ consists of exactly k compact components. (See 3.3.) Let K_x be that component that contains x. There exists an elementary region R such that

$$(1) \qquad K_x \subset R \subset \operatorname{cl} R \subset X - f^{-1}(D) \quad \text{and} \quad (f^{-1}f(x)) \cdot \operatorname{cl} R = K_x.$$

It will be shown in 3.7 that $\mu(x, f, R)$ is independent of the particular choice of R provided that R satisfies (1). On that basis the following may be defined:

$$\mu(x) \equiv \mu(f(x), f, R).$$

(Because of the nature of point inverses of f, μ is a point function version of the function defined in II. 3.4 of [7].)

3.7. *$\mu(x, f, R_1) = \mu(x, f, R_2)$ provided that R_1 and R_2 satisfy* (1).

PROOF. Let Q be the component of $R_1 \cdot R_2$ that contains K_x. There exists an elementary region R such that

$$K_x \subset R \subset \operatorname{cl} R \subset Q \subset R_1 \cdot R_2.$$

From 1.4, $\mu(f(x), f, R_1) = \mu(f(x), f, R) = \mu(f(x), f, R_2)$.

Use will be made of the following in investigating the function μ.

3.8. YOUNG'S MODIFICATION THEOREM. (See [6].) *Let N be a closed 2-cell with interior G and boundary J. Suppose X is a continuum and $m: X \to N$ is a monotone mapping of X onto N such that $m^{-1}(G)$ is an open 2-cell. Then there is a mapping $h: X \to N$ of X onto N such that $h(x) = m(x)$ for $x \in m^{-1}(J)$ and $h^{-1}h(x) = x$ for $x \in m^{-1}(G)$.*

3.9. *$|\mu(x)| = 1$ for each $x \in X - f^{-1}(D)$.*

PROOF. Let $x \in X - f^{-1}(D)$. Let R be an elementary region admissible for the computation of $\mu(x)$. (See 3.6.) Let $y = m(x) \in M$. Note that l is a local homeomorphism at y and recall that m is compact.

Hence by 2.5 there exists a closed 2-cell neighborhood N of y such that $m^{-1}(N)\subset R$ and such that l is a homeomorphism on N. Let $G=m^{-1}(\text{int } N)$. By 2.4 G is an open 2-cell. Thus 3.8 can be applied to $m|m^{-1}(N)$ and using this a modification $h: X\to M$ of m can be defined such that $h(z)=m(z)$ for $z\in X-G$ and h is one-to-one on G. Let z^* be the unique point in G such that $h(z^*)=y$. Note also $h^{-1}(y)\cdot R=z^*$. Let W be a closed 2-cell such that $z^*\in\text{int } W\subset W\subset G$. Note that $lh|W$ is a homeomorphism. Then by 1.6 and from the fact that $lh(z^*)=l(y)=f(x)$, it follows that $\mu(f(x), lh, \text{int } W)=\pm 1$. Further since $[(lh)^{-1}f(x)]\cdot R=z^*$, it follows from 1.4 that $\mu(f(x), lh, \text{int } W)=\mu(f(x), lh, R)$. Moreover by 1.3, since $f=lh$ on $\text{Fr}(R)$, $\mu(f(x), f, R)=\mu(f(x), lh, R)$. Combining this with the two previous equalities involving μ yields $\mu(f(x), f, R)=\pm 1$.

3.10. *μ is continuous on $X-f^{-1}(D)$.* (Hence by 3.5 $\mu(x)$ is constant on $X-f^{-1}(D)$.)

Proof. Let $x\in X-f^{-1}(D)$ and K_x the component of $f^{-1}(x)$ that contains x. Choose R as in (1) of 3.6 so that $\mu(x)=\mu(f(x), f, R)$. Choose a closed 2-cell N such that $f(x)\in\text{int } N\subset\text{cl } N\subset f(X)-D-f(\text{Fr}(R))$ and is chosen as in 3.4. Let Q be the component of $f^{-1}(\text{int } N)$ that contains K_x. Since $f(Q_i)=\text{int } N$ for each component Q_i of $f^{-1}(\text{int } N)$ (see 3.4) and since $f^{-1}(x)\cdot R=K_x$, it follows that Q is the only component of $f^{-1}(\text{int } N)$ that intersects R.

Now let $y\in Q$. Let K_y be the component of $f^{-1}(f(y))$ that contains y. However from the way in which N was chosen (see 3.4) K_y is the only component of $f^{-1}f(y)$ that intersects R. Thus R is admissible for the computation of $\mu(y)$ and $\mu(y)=\mu(f(y), f, R)$. However by 1.2, $\mu(f(y), f, R)=\mu(f(x), f, R)$ and thus $\mu(x)=\mu(y)$.

3.11. *Let $f: X\to P$ be as in* 3.1. *Then f is minimal.*

Proof. Let N be a closed 2-cell in X. From 1.5 and 1.7 we need show only that $\mu(x, f, \text{int } N)\neq 0$ for each $x\in f(N)-f(\text{Fr } N)$. Let $x\in f(N)-f(\text{Fr } N)$. Define D as in 3.3. Let U_x be the component of $P-f(\text{Fr } N)$ that contains x. There exists a point $y\in U_x\cdot[f(\text{int } N)-D]$. This follows from the quasi-openness of f and the definition of D. By 1.2, $\mu(y, f, \text{int } N)=\mu(x, f, \text{int } N)$. Let $K_1, K_2, \cdots, K_h$ be the components of $f^{-1}(y)$ that intersect and hence are contained in int N. There exist elementary regions $R_1, R_2, \cdots, R_h$ such $K_i\subset R_i\subset\text{cl } R_i\subset\text{int } N$ and $R_i\cdot R_j=\emptyset$ if $i\neq j$. By 1.4, $\mu(y, f, \text{int } N)=\sum_i \mu(y, f, R_i)$. Let $x_i\in K_i$. Thus, from definition of μ, 3.9, and 3.10 $|\sum_i \mu(y, f, R_i)|=|\sum_i \mu(x_i)|=h$ and $\mu(x, f, \text{int } N)=\mu(y, f, \text{int } N)\neq 0$.

Theorem 1.1. *The validity of the Theorem* 1.1 *stated in the introduction now follows from* 2.6, 3.2 *and* 3.11.

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