ON EXPONENTIALLY CLOSED FIELDS

NORMAN L. ALLING

It is well known [4] that the non-Archimedean residue class fields \( K \) of the ring of continuous real valued functions on a space are real-closed and \( \eta \)-sets. It does not appear to be known that the exponential function in the reals induces an exponential function in \( K \) (definitions to follow); thus \( K \) is exponentially closed. The property of being exponentially closed is a new invariant which will be applied to totally ordered fields in this paper.

A totally ordered field \( K \) will be called \textit{exponentially closed} if (i) there exists an order preserving isomorphism \( f \) of the additive group of \( K \) onto \( K^+ \), the multiplicative group of positive elements of \( K \), and (ii) there exists a positive integer \( n \) such that \( 1 + 1/n < f(1) < n \); such an isomorphism will be called an \textit{exponential function} in \( K \).

In §0 Archimedean exponentially closed fields will be considered, the rest of the paper being devoted to the non-Archimedean case. In §1 some necessary conditions for a non-Archimedean field to be exponentially closed will be given, followed in §2 by some examples. In §3 a set of sufficient conditions will be given, followed by an example.

A totally ordered field \( K \) will be called \textit{root-closed} if \( K^+ \) is divisible. Clearly exponentially closed fields and real-closed fields are root-closed.

0. An Archimedean totally ordered field is isomorphic to a unique subfield of the reals. Let \( K \) be an exponentially closed subfield of the reals and let \( f \) be an exponential function in \( K \). If \( a = f(1) \) then \( f(x) = a^x \) for all \( x \in K \). Conversely, if \( a \in K \), \( a > 1 \), and if \( g(x) \) is defined to be \( a^x \) for all \( x \in K \), then \( g \) is an exponential function in \( K \). Thus, any subfield of the reals is contained in a unique exponentially closed subfield of the reals, both having the same cardinality. The field of real algebraic numbers is, by definition, real-closed. However \( 2^{1/\alpha} \) is not in it, hence it is not exponentially closed. It is not known to the author if exponentially closed fields need be real-closed.

1. \textbf{Necessary conditions.} Let \( K \) be a non-Archimedean field. It is well known [3] that one can associate with \( K \) a totally ordered group

\[ \text{(Presented to the Society, April 14, 1961; received by the editors April 17, 1961 and, in revised form, July 28, 1961.\textsuperscript{1}} \]
G and a homomorphism \( V \) of the multiplicative group of \( K \) onto \( G \) satisfying the following conditions: (1) \( V \) is order preserving on \( K^+ \), (2) \( V(a \pm b) \leq \max(V(a), V(b)) \) (\( V(0) \) being the symbol \(-\infty\) treated in the usual way), and (3) \( V(a) = V(b) \) if and only if there exists a positive integer \( n \) such that \( |a| \leq n|b| \) and \( |b| \leq n|a| \). The mapping \( V \) will be called a natural valuation on \( K \); clearly any two such mappings are essentially identical. The valuation ring of \( V \) is \( \mathcal{O} = \{ a \in K : V(a) \leq 0 \} \) and its maximal ideal \( P = \{ a \in K : V(a) < 0 \} \). Clearly the residue class field of \( K \), \( O/P = k \), is an Archimedean field.

Assume, in addition, that \( K \) is exponentially closed and that \( f \) is an exponential function in \( K \).

**Lemma 1.1.** The restriction of \( f \) to \( O \) maps \( O \) onto the group of positive units of \( O \). Further, \( a \in P \) if and only if \( f(a) - 1 \in P \).

**Proof.** Since \( f(1) < n \), \( f \) maps \( O \) into the positive units of \( O \). Let \( a \) be a positive unit of \( O \). There exists \( m \in \mathbb{N} \), the set of positive integers, such that \( 1/m < a < m \). Let \( b = f^{-1}(a) \). It suffices to show that \( b \in O \). There exists \( i \in \mathbb{N} \) such that \( (1+1/n)^i > m \). Thus \( f(i) = f(1)^i > (1+1/n)^i > m > f(b) \), and \( i > b \). Since \( 1+1/n < f(1) \), \( f(-1) < n/(n+1) \). Since \( n/(n+1) < 1 \), there exists \( t \in \mathbb{N} \) such that \( (n/(n+1))^t < 1/m \). Thus \( f(-t) = f(-1)^t < (n/(n+1))^t < 1/n < f(b) \) and \( -t < b \), proving that \( b \in O \) and hence the first assertion is proved.

Let \( h \) be the canonical homomorphism of \( O \) onto \( k \). Clearly \( r = hf \) is an order preserving homomorphism of \( O \) onto the multiplicative group of positive units of \( k \). Clearly \( a \in P \) if and only if \( -1 < ma < 1 \) for all integers \( m \). By condition (ii), \( 1+1/n \leq r(1) \leq n \) and \( 1/n \leq r(-1) \leq n/(n+1) \). Hence \( a \) is in \( P \) if and only if \( 1/n \leq (r(a))^m \leq n \) for all integers \( m \): i.e., \( r(1) = 1 \) or equivalently \( f(a) - 1 \in P \), proving the lemma.

The following theorem is an immediate consequence of this lemma.

**Theorem 1.2.** The residue class field of a non-Archimedean exponentially closed field is an Archimedean exponentially closed field.

The restriction of \( V \) to \( K^+ \) is an order preserving homomorphism onto \( G \) whose kernel is the group of positive units of \( O \); thus \( Vf \) is an order preserving homomorphism of the additive group of \( K \) onto \( G \) whose kernel is \( O \), proving the following theorem.

**Theorem 1.3.** If \( K \) is a non-Archimedean exponentially closed field whose valuation ring is \( O \) and whose value group is \( G \) then there exists an order preserving group isomorphism that sends \( K/O \) onto \( G \).

It is well known [3] that given a totally ordered Abelian group \( G \)
there exists a mapping \( W \) of \( G \) onto a totally ordered set that has all the properties of \( V \), except that of being a homomorphism. Such a mapping, characterized by these properties, will be called a natural valuation on \( G \). Let \( G^+ \) be the set of positive elements of \( G \). Then \( S = W(G^+) \) will be called the value set of \( G \). For \( s \in S \) let \( G_s = \{ g \in G : W(g) \leq s \} / \{ g \in G : W(g) < s \} \). Clearly \( G_s \), which will be referred to as the factor of \( G \) associated with \( s \), is an Archimedean group.

**Corollary 1.4.** Assume that \( K \) is a non-Archimedean exponentially closed field. Let \( G \) be the value group of \( K \) and \( k \) the residue class field of \( K \). Then \( G^+ \) is isomorphic as an ordered set to \( W(G^+) \) and the factors of \( G \) are isomorphic to \( k \).

**Proof.** By Theorem 1.3, \( K/O \) and \( G \) are isomorphic; thus they have isomorphic value sets. The value set of \( K/O \) under the natural valuation induced by \( V \) is \( G^+ \), proving the first assertion. Let \( g \in G^+ \). The factor of \( K/O \) associated with \( g \) is isomorphic to the factor \( K_o \) of \( K \) associated with \( g \). Let \( a \in K \) such that \( V(a) = g \). Then \( K_o = Oa/Pa \), which is isomorphic to \( O/P = k \), proving the corollary.

2. **Examples.** Under pointwise operations, the set \( C(X) \) of all continuous functions from a completely regular Hausdorff space into the reals is a lattice-ordered ring. If \( a \in C(X) \) then \( e^a \in C(X) \); further, \( a \) and \( e^a - 1 \) have the same zeros and hence [4] belong to the same maximal ideals. Let \( K \) be a non-Archimedean residue class field of \( C(X) \) [4] and let \( h \) be the associated canonical homomorphism. For \( a' \in K \) let \( a \in h^{-1}(a') \), and let \( f(a') = h(e^a) \). Since \( a' = 0 \) if and only if \( h(e^a - 1) = 0 \), \( f \) is a well defined isomorphism of \( K \) into \( K^+ \). Since \( h \) and \( a \rightarrow e^a \) are order preserving, so then is \( f \). For \( a' \geq 1 \) we may choose \( a \geq 1 \). Let \( b = \log a \) and let \( b' = h(b) \). Clearly \( f(b') = a' \). For \( 0 < a' < 1 \) we may apply the argument above to \( 1/a' \); thus \( K \) is exponentially closed.\(^2\)

It is well known [4] that such fields are real-closed, have the reals as their residue class field and are \( \eta \)-sets in the sense of the following definition. Let \( \alpha \) be an ordinal number and let \( T \) be a totally ordered set. \( T \) is called an \( \eta_{\alpha} \)-set if, given subsets \( A \) and \( B \) of \( T \) of power less than \( \aleph_\alpha \) such that \( A < B \), then there exists \( t \in T \) such that \( A < \{ t \} < B \).

It has been shown [2] that if \( \alpha > 0 \), \( \aleph_\alpha \) is a regular cardinal number, and \( \sum_{\alpha < \alpha} 2^{\aleph_\alpha} \leq \aleph_\alpha \), then a real-closed field exists that is an \( \eta_{\alpha} \)-set of power \( \aleph_\alpha \). Let \( K \) be such a field. Clearly \( K \) is non-Archimedean. Let \( f_\alpha(n) = 2^n \) for all integers \( n \). Both the additive group of \( K \) and the

\(^2\) According to Henriksen, this argument can be used to show that the residue class fields of uniformly closed phi-algebras are exponentially closed.
multiplicative group of positive elements of $K$ are totally ordered
Abelian divisible groups that are $\eta_{\alpha}$-sets of power $\mathbb{N}_1$. Thus by Theo-
rem B [1] $f_0$ extends to an exponential function in $K$, proving that
$K$ is exponentially closed.

Let $k$ be an Archimedean field and let $T$ be a nonempty totally
ordered set. For $a \in k^T$ let $s(a) = \{ t \in T : a(t) \neq 0 \}$. A subset of $T$ is
called anti-wellordered if every nonempty subset of it has a greatest
element. Let $k\{ T \}$ be defined to be $\{ a \in k^T : s(a) \text{ is anti-wellordered} \}$. Clearly $k\{ T \}$ is an Abelian group under pointwise addition. For
$a \in k\{ T \}, a \neq 0$, let $d(a)$ be the greatest element in $s(a)$. Define $a > 0$
if $a(d(a)) > 0$; then $k\{ T \}$ is a totally ordered group, $d$ is a natural
valuation and $T$ is its value set.

For an ordinal number $\alpha$ let $k\{ T \}_\alpha = \{ a \in k\{ T \} : \text{the cardinal num-
ber of } s(a) \text{ is less than } \aleph_{\alpha} \}$. Clearly $k\{ T \}_\alpha$ is a subgroup of $k\{ T \}$. Let $G$ be a nonzero totally ordered Abelian group. For $a, b \in k\{ G \}$ let $(ab)(g) = \sum_{x \in G} a(x)b(g-x)$. It is well known [5] that, under this
multiplication, $k\{ G \}$ is a totally ordered field. Let $\alpha$ be a nonzero
ordinal number; then $k\{ G \}_\alpha$ is a subfield of $k\{ G \}$. Further, $d$ re-
stricted to $k\{ G \}_\alpha$ is a natural valuation of $k\{ G \}_\alpha$, its value group
being $G$ and its residue class field $K$.

Let $G$ be a totally ordered Abelian divisible group that is an $\eta_\alpha$-set
of power $\mathbb{N}_1$ and let $K = R\{ G \}$. It was shown in [2] that $K$ is a real-
closed field that is an $\eta_\alpha$-set and has as its residue class field the reals;
thus $K$ might be conjectured to be isomorphic to a residue class field
of $C(X)$ for some $X$. However $K/O$ is isomorphic to $R\{ G^+ \}$ which is
of power $2^{\mathbb{N}_1}$, whereas $G$ is of power $\aleph_\alpha$; thus, by Theorem 1.3, $K$ is
not exponentially closed and hence not isomorphic to any residue class
field of $C(X)$ for any space $X$.

3. Sufficient conditions. Let $k$ be an Archimedean field, $\alpha$ a non-
zero ordinal, $G$ a nonzero totally ordered Abelian group, and let
$K = k\{ G \}_\alpha$. The valuation ideal of $K$ is $k\{ G^- \}$, $G^-$ being the set of
all negative elements of $G$. It has been shown [5] that given a nonzero
element $q$ of $P$ then the semigroup $\omega_\alpha(q) = \bigcup_{n \in \mathbb{N}} ns(q)$ of $G$ is anti-
wellder, and further given $g$ in it there exists $m \in \mathbb{N}$ such that
g $\in \bigcup_{n \in \mathbb{N}} ns(q)$. Thus given a sequence $(a_n)_{n \in \mathbb{N}}$ in $k$, $r = \sum_{n=1}^{\infty} a_nq^n$ is a
well defined element of $P$. Further, given $b \in K$, $rb = \sum_{n=1}^{\infty} a_nb^n$.

For $q \in P$ let $\exp q = \sum_{n=0}^{\infty} q^n/n!$ and let $\log 1 + q = \sum_{n=1}^{\infty} (-1)^{n-1}q^n/n$. By direct calculation it is seen that for all
$q \in P, r \in P$, $\exp q \exp r = \exp q + r$. From analysis we know that
$\sum_{n=1}^{\infty} (-1)^{n-1}(\sum_{m=1}^{n} x^m/m!) n/n$ converges for all real $x$ such that
$|x| < \log 2$; and further that the sum of this series, since it is the ex-
pansion of \( \log e^x \), is \( x \). Hence the coefficients of this series are the same as the coefficients of the power series \( x \). Thus \( \log \exp q = q \) for all \( q \in P \), proving that \( \exp \) maps \( P \) onto \( 1+P \) and is one-to-one.

Let \( K \) be a non-Archimedean field with value group \( G \) and residue class field \( k \). We will say that \( K \) is properly imbedded in \( k \{ G \} \) if it is imbedded in \( k \{ G \} \) such that given \( a \in K \), \( V(a) = \delta(a) \), and such that \( k \{ G \} \subset K \). Generalizing somewhat a well known result stated by Conrad [3, p. 328] we get the following: if \( K \) is real-closed it can be properly imbedded in \( k \{ G \} \).

**Theorem 3.1.** A non-Archimedean field \( K \) with valuation ring \( O \), valuation ideal \( P \), value group \( G \) and residue class field \( k \) is exponentially closed if the following hold: (0) \( K \) is root-closed, (1) \( K \) is exponentially closed, (2) \( K/O \) is order isomorphic to \( G \), and (3) \( K \) may be properly imbedded in \( k \{ G \} \) in such a way that if \( q \in P \) then \( \exp q \) and \( \log 1+q \) \( \in K \).

**Proof.** Let \( K \) be imbedded in \( k \{ G \} \) such that condition (3) holds; thus \( \exp \) is an order preserving isomorphism of \( P \) onto the multiplicative group of \( 1+P \). Let \( k = k1 \). Clearly the ring \( O \) is the direct sum of \( k \) and \( P \), the order on the sum being lexicographic. By condition (1), \( k \) is exponentially closed; thus given \( a \in k \), \( a > 1 \), the mapping \( x \rightarrow a^x \) is an exponential in \( k \). For \( y \in O \) let \( y = x + q \), \( x \in k \) and \( q \in P \), this decomposition being unique. Let \( f_a(y) = a^y \exp q \). Clearly \( f_a \) is an order preserving isomorphism of \( O \) onto the group of positive units of \( O \). The additive group of \( K \) is the direct sum of \( K/O \) and \( O \), the order in the sum being lexicographic. An element \( u \) in \( K \) can be expressed uniquely as \( z+y \), \( z \in K/O \) and \( y \in O \). By condition (2) there exists an order preserving isomorphism \( t \) of \( K/O \) onto \( G \). Let \( f(u) = (t(z), f_0(y)) \). The valuation \( V \), restricted to \( K^+ \), is an order preserving homomorphism of the multiplicative group of \( K^+ \) (which is divisible by condition (0)) onto \( G \) whose kernel is the group of positive units of \( O \). Thus the totally ordered group \( K^+ \) is the direct product of \( G \) and the group of multiplicative units of \( O \), the order being lexicographic. Hence \( f \) becomes an exponential function of \( K \), proving that \( K \) is exponentially closed, proving the theorem.

**Note.** Conditions (0), (1) and (2) are necessary for \( K \) to be exponentially closed.

Let \( E \) be an \( \eta \)-set of power \( \aleph_1 \) and let \( (x_n)_{n \in N} \) be a strictly increasing sequence in \( E \). Let \( E_n = \{ x \in E : x < x_n \} \). Then \( E_n \) is an \( \eta \)-set of power \( \aleph_1 \). Let \( E' = \bigcup_{n \in N} E_n \). Since \( E' \) has a countable cofinal sequence it is not an \( \eta \)-set. Let \( G = R \{ E' \} \) and let \( G_n = R \{ E_n \} \). Then \( G_n \) is an Abelian divisible group that is an \( \eta \)-set of power \( \aleph_1 \) [2]. Further,
$G = \bigcup_{n \in \mathbb{N}} G_n$; thus $G^+$ is order isomorphic to $E'$ which, under the natural valuation $d$, is the value set of $G$ (cf. Corollary 1.4).

$K = R \{G\}_1$ is a real-closed field (hence a root-closed field) that has the reals as its residue class field; thus $K$ satisfies conditions (0) and (1). $K/O$ is isomorphic to $R \{G^+\}_1$ which, since $G^+$ is isomorphic to $E'$, is isomorphic to $R \{E'\}_1$: i.e., to $G$; thus $K$ satisfies condition (2). Clearly condition (3) holds. Thus, by Theorem 3.1, $K$ is exponentially closed. However, since $K$ has a countable cofinal sequence it is not an $\eta_1$-set.

**Bibliography**