THE OPERATION OF THE UNIVERSAL DOMAIN ON THE PLANE[1]

SHERWOOD EBEY

The problem we are considering is if

$$(1) \qquad (t, x, y) \to (P(t, x, y),\ Q(t, x, y))$$

defines a regular operation of the additive group of the universal domain Ω on the affine plane A^2, then what can be said about the form of the polynomials P and Q?

First we note that any change of coordinates for A^2 is defined by equations

$$(2) \qquad \begin{aligned} u &= M(x, y), & x &= R(u, v), \\ v &= N(x, y), & y &= S(u, v), \end{aligned}$$

where M, N, R, S are polynomials with coefficients in Ω. Now the operation of Ω on A^2 given by equations (1) is defined in uv-coordinates by

$$(3) \quad (t, u, v) \to (M(P(t, R, S), Q(t, R, S)),\ N(P(t, R, S), Q(t, R, S))).$$

We shall use a theorem from a paper by Engel [1] which states that in a change of coordinates such as (2) the degree of M must divide the degree of N or vice versa. Engel's proof is for characteristic $=0$. Assuming this result we will prove the following theorem.

THEOREM. *If the characteristic of Ω is 0 and if there is a regular operation of Ω on A^2, then there is a change of coordinates of A^2 such that in terms of the uv-coordinates the given operation has the form*

$$(t, u, v) \to (u, v + tf(u)) \qquad \textit{with } f \in \Omega[u].$$

We will use the theory of algebraic groups as developed by Rosenlicht in [2]. If $H \in \Omega(x, y)$ and $t \in \Omega$, then $\lambda_t H$ is the function $H(P(t, x, y), Q(t, x, y))$. H is called *invariant* if $\lambda_t H = H$ for all $t \in \Omega$. Now to prove the theorem we proceed by proving a sequence of three lemmas.

LEMMA 1. *There exists a nonconstant polynomial in $\Omega[x, y]$ which is invariant.*

PROOF. Let k be an algebraically closed field of definition for Ω, A^2,

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and the operation on A^2. Consider the variety W of Ω-orbits on A^2. If this were of dimension 0, any two independent generic points for A^2 over k would belong to the same orbit. This is impossible and therefore $\dim(W)>0$.

Now $\dim(W)>0$ implies that the field $k(W)$ has a nonconstant function. But $k(W)$ is k-isomorphic to the subfield of invariant functions in $k(x, y)$. Hence there exists a nonconstant function H in $k(x, y)$ which is invariant.

Let $H=H_1/H_2$ where $H_i \in k[x, y]$ and H_1 and H_2 are relatively prime. Let t be a variable over $k(x, y)$. Now $\lambda_t H=H$ implies $\lambda_t H_1/\lambda_t H_2 =H_1/H_2$. Using the unique factorization of $k[t, x, y]$ it can be shown that $\lambda_t H_i=H_i$. Since H is a nonconstant function, either H_1 or H_2 is the required nonconstant invariant polynomial in $k[x, y]$. Q.e.d.

Note that by taking powers of such an invariant polynomial we can obtain an invariant polynomial of arbitrarily high degree.

For the operation defined by polynomials (1) we must have $P(0, x, y)=x$ and $Q(0, x, y)=y$. Therefore these polynomials must be of the form

$$\begin{aligned} P(t, x, y) &= x + tf_1(x, y) + t^2f_2(x, y) + \cdots + t^nf_n(x, y), \\ Q(t, x, y) &= y + tg_1(x, y) + t^2g_2(x, y) + \cdots + t^mg_m(x, y). \end{aligned} \tag{4}$$

LEMMA 2. *If the polynomials P and Q which define the operation of Ω on A^2 are such that $f_n\neq 0$, $g_m\neq 0$, $0<n\leqq m$, and $m/n=l/j$ with $(l, j)=1$, then $f_n^l=cg_m^j$ where c is a constant.*

PROOF. Let the weight of a monomial $x^\nu y^\mu$ be defined as the integer $n\nu+m\mu$. The weight of a polynomial will be the maximum of the weights of its monomials. Now

$$\lambda_t(x^\nu y^\mu) = P^\nu Q^\mu = (f_n^\nu g_m^\mu)t^{n\nu+m\mu} + \text{terms of lower degree in } t.$$

By Lemma 1 there is a nonconstant polynomial $H(x, y)$ such that $\lambda_t H=H$. If s is the weight of H and if $x^\nu y^\mu$ is the monomial of H of weight s and with degree in x maximal, then the part of H of weight s must be of the form

$$H_s(x, y) = a_0x^\nu y^\mu + a_1x^{\nu-l}y^{\mu+j} + a_2x^{\nu-2l}y^{\mu+2j} + \cdots + a_qx^{\nu-ql}y^{\mu+qj}.$$

The coefficient of t^s in $\lambda_t H$ will be $H_s(f_n, g_m)$ which must be identically zero since $\lambda_t H=H$. From this we obtain a relation of the form

$$a_0f_n^{ql} + a_1f_n^{(q-1)l}g_m^j + \cdots + a_qg_m^{qj} = 0.$$

This implies that f_n^l/g_m^j must equal one of the roots of an algebraic equation of degree q. Therefore $f_n^l = cg_m^j$ where c is some constant. Q.e.d.

LEMMA 3. *If the polynomials P and Q in* (4) *are such that* $f_n \neq 0$, $g_m \neq 0$, $0 < n \leqq m$, *then* $n | m$.

PROOF. Choose an invariant polynomial $H(x, y)$ so that the degree of H is greater than the maximum of the degrees of

$$\{f_1, f_2, \cdots, f_n, g_1, g_2, \cdots, g_m\}.$$

Let ν be the degree of H.

Lemma 2 implies that if f_n and g_m are nonconstant polynomials, then $g_m = G^l(x, y)$ and $f_n = cG^j(x, y)$ where c is a constant and l and j are as in Lemma 2. Let μ be the degree of G. (If f_n and g_m are constants, $\mu = 0$.)

Now since $(t, x, y) \to (P(t, x, y), Q(t, x, y))$ defines a regular operation of Ω on A^2, it follows that

$$\Omega[t, x, y] = \Omega[t, P(t, x, y), Q(t, x, y)]. \tag{5}$$

If we let $t = H(x, y)$ in (5) we have that

$$\Omega[x, y] = \Omega[H, x, y] = \Omega[H, P(H, x, y), Q(H, x, y)]. \tag{6}$$

Since H is invariant $H(x, y) = H(P(H, x, y), Q(H, x, y))$. Thus (6) implies

$$\Omega[x, y] = \Omega[P(H, x, y), Q(H, x, y)]. \tag{7}$$

From the choice of ν it follows that the degree of the first generator on the right side of (7) must be the degree of $H^n f_n$ which is $n\nu + j\mu$. Similarly the degree of the second generator on the right side of (7) is the degree of $H^m g_m$ which is $m\nu + l\mu$.

By the result in [1] that we referred to above, $(n\nu + j\mu) | (m\nu + l\mu)$. This implies that $j | l$. But since $(j, l) = 1$, $j = 1$ and $n | m$. Q.e.d.

We will now prove the theorem. Suppose we have polynomials P and Q of form (4) with $n \neq 0$ and $m \neq 0$. Let $n \leqq m$. By Lemma 3 we know that $n | m$ so that in the notation of Lemma 2, $m/n = l$ and $j = 1$. Thus Lemma 2 implies that $g_m = cf_n^l$ where c is a constant.

Make the following change of coordinates for A^2:

$$u = x, \qquad x = u,$$
$$v = y - cx^l, \qquad y = v + cu^l.$$

Applying equation (3) to this particular change of coordinates, the given operation will be defined in uv-coordinates by

$$(8) \quad (t, u, v) \to (P(t, u, v + cu^l), Q(t, u, v + cu^l) - cP^l(t, u, v + cu^l)).$$

The coefficient of t^m in $Q(t, u, v+cu^l) - cP^l(t, u, v+cu^l)$ is $g_m(u, v+cu^l) - cf_n^l(u, v+cu^l)$, which is zero.

Hence (8) reduces to an expression of form (4) in which the degree of P in t is n while the degree of Q in t is less than m. We can repeat this process until we obtain coordinates in which either P or Q will have degree 0 in t.

Thus for any regular operation of Ω on A^2, there are coordinates u, v so that the operation has the form

$$(t, u, v) \to (u, Q(t, u, v)).$$

But we may consider $(t, v) \to (Q(t, u, v))$ as defining an operation on the line. Now Q must satisfy the identity

$$Q(s + t, u, v) = Q(s, u, Q(t, u, v)).$$

Since characteristic $=0$, it follows that $Q(t, u, v)$ must be of the form $v+tf(u)$ where $f \in \Omega[u]$. Thus in uv-coordinates the given operation has the form

$$(t, u, v) \to (u, v + tf(u)).$$

This completes the proof of the theorem.

References

1. W. Engel, *Ganze Cremona-Transformationen von Primzahlgrad in der Ebene*, Math. Ann **136** (1958), 319–325.

2. M. Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443.

Wheaton College, Wheaton, Illinois