THE OPERATION OF THE UNIVERSAL
DOMAIN ON THE PLANE

SHERWOOD EBEY

The problem we are considering is if

\[ (t, x, y) \rightarrow (P(t, x, y), Q(t, x, y)) \]

defines a regular operation of the additive group of the universal
domain \( \Omega \) on the affine plane \( A^2 \), then what can be said about the form
of the polynomials \( P \) and \( Q \)?

First we note that any change of coordinates for \( A^2 \) is defined by
equations

\[ \begin{align*}
    u &= M(x, y), & x &= R(u, v), \\
    v &= N(x, y), & y &= S(u, v),
\end{align*} \]

where \( M, N, R, S \) are polynomials with coefficients in \( \Omega \). Now the
operation of \( \Omega \) on \( A^2 \) given by equations (1) is defined in \( uv \)-coordinates by

\[ (t, u, v) \rightarrow (M(P(t, R, S), Q(t, R, S)), N(P(t, R, S), Q(t, R, S))). \]

We shall use a theorem from a paper by Engel [1] which states that
in a change of coordinates such as (2) the degree of \( M \) must divide the
degree of \( N \) or vice versa. Engel's proof is for characteristic \( = 0 \).
Assuming this result we will prove the following theorem.

**Theorem.** If the characteristic of \( \Omega \) is 0 and if there is a regular
operation of \( \Omega \) on \( A^2 \), then there is a change of coordinates of \( A^2 \) such
that in terms of the \( uv \)-coordinates the given operation has the form

\[ (t, u, v) \rightarrow (u, v + tf(u)) \] with \( f \in \Omega[u] \).

We will use the theory of algebraic groups as developed by Rosen-
licht in [2]. If \( H \in \Omega(x, y) \) and \( t \in \Omega \), then \( \lambda H \) is the function
\( H(P(t, x, y), Q(t, x, y)) \). \( H \) is called *invariant* if \( \lambda H = H \) for all \( t \in \Omega \).
Now to prove the theorem we proceed by proving a sequence of
three lemmas.

**Lemma 1.** There exists a nonconstant polynomial in \( \Omega[x, y] \) which is
invariant.

**Proof.** Let \( k \) be an algebraically closed field of definition for \( \Omega, A^2 \),

Received by the editors September 1, 1961.

1 This paper is based on a portion of the author's doctoral dissertation, written at Northwestern University under the supervision of Professor M. Rosenlicht.

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and the operation on $A^2$. Consider the variety $W$ of $\Omega$-orbits on $A^2$. If this were of dimension 0, any two independent generic points for $A^2$ over $k$ would belong to the same orbit. This is impossible and therefore $\dim(W) > 0$.

Now $\dim(W) > 0$ implies that the field $k(W)$ has a nonconstant function. But $k(W)$ is $k$-isomorphic to the subfield of invariant functions in $k(x, y)$. Hence there exists a nonconstant function $H$ in $k(x, y)$ which is invariant.

Let $H = H_1/H_2$ where $H_1 \subseteq k[x, y]$ and $H_1$ and $H_2$ are relatively prime. Let $t$ be a variable over $k(x, y)$. Now $\lambda_i H = H$ implies $\lambda_i H_1/\lambda_i H_2 = H_1/H_2$. Using the unique factorization of $k[t, x, y]$ it can be shown that $\lambda_i H = H$. Since $H$ is a nonconstant function, either $H_1$ or $H_2$ is the required nonconstant invariant polynomial in $k[x, y]$. Q.e.d.

Note that by taking powers of such an invariant polynomial we can obtain an invariant polynomial of arbitrarily high degree.

For the operation defined by polynomials (1) we must have $P(0, x, y) = x$ and $Q(0, x, y) = y$. Therefore these polynomials must be of the form

$$P(t, x, y) = x + tf_1(x, y) + t^2f_2(x, y) + \cdots + t^nf_n(x, y),$$

$$Q(t, x, y) = y + tg_1(x, y) + t^2g_2(x, y) + \cdots + t^mg_m(x, y).$$

**Lemma 2.** If the polynomials $P$ and $Q$ which define the operation of $\Omega$ on $A^2$ are such that $f_n \neq 0$, $g_m \neq 0$, $0 < n \leq m$, and $m/n = l/j$ with $(l, j) = 1$, then $f_n = c g_m$ where $c$ is a constant.

**Proof.** Let the weight of a monomial $x^s y^t$ be defined as the integer $nv + mu$. The weight of a polynomial will be the maximum of the weights of its monomials. Now

$$\lambda_i(x^s y^t) = P_i Q_i = (f_n g_m)^{n/m + m} + \text{terms of lower degree in } t.$$ 

By Lemma 1 there is a nonconstant polynomial $H(x, y)$ such that $\lambda_i H = H$. If $s$ is the weight of $H$ and if $x^s y^t$ is the monomial of $H$ of weight $s$ and with degree in $x$ maximal, then the part of $H$ of weight $s$ must be of the form

$$H_s(x, y) = a_0 x^s y^t + a_1 x^{s-1} y^{t+1} + a_2 x^{s-2} y^{t+2} + \cdots + a_s x^{s-q} y^{t+q}.$$ 

The coefficient of $t^j$ in $\lambda_i H$ will be $H_s(f_n, g_m)$ which must be identically zero since $\lambda_i H = H$. From this we obtain a relation of the form

$$a_0 f_n^q + a_1 f_n^{(q-1)} g_m + \cdots + a_s g_m^q = 0.$$ 

This implies that $f_n^j/g_m^j$ must equal one of the roots of an algebraic
equation of degree $q$. Therefore $f_n' = cg_m$ where $c$ is some constant.
Q.e.d.

**Lemma 3.** If the polynomials $P$ and $Q$ in (4) are such that $f_n \neq 0$, $g_m \neq 0$, $0 < n \leq m$, then $n | m$.

**Proof.** Choose an invariant polynomial $H(x, y)$ so that the degree of $H$ is greater than the maximum of the degrees of $\{f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m\}$.

Let $v$ be the degree of $H$.

Lemma 2 implies that if $f$ and $g$ are nonconstant polynomials, then $g_m = G^l(x, y)$ and $f_n = cG^l(x, y)$ where $c$ is a constant and $l$ and $j$ are as in Lemma 2. Let $\mu$ be the degree of $G$. (If $f_n$ and $g_m$ are constants, $\mu = 0$.)

Now since $(t, x, y) \rightarrow (P(t, x, y), Q(t, x, y))$ defines a regular operation of $\Omega$ on $A^2$, it follows that

$$\Omega[t, x, y] = \Omega[t, P(t, x, y), Q(t, x, y)].$$

If we let $t = H(x, y)$ in (5) we have that

$$\Omega[x, y] = \Omega[H, x, y] = \Omega[H, P(H, x, y), Q(H, x, y)].$$

Since $H$ is invariant $H(x, y) = H(P(H, x, y), Q(H, x, y))$. Thus (6) implies

$$\Omega[x, y] = \Omega[P(H, x, y), Q(H, x, y)].$$

From the choice of $\nu$ it follows that the degree of the first generator on the right side of (7) must be the degree of $H^*f_n$, which is $nv + j\mu$. Similarly the degree of the second generator on the right side of (7) is the degree of $H^*g_m$ which is $mv + l\mu$.

By the result in [1] that we referred to above, $(nv + j\mu) \mid (mv + l\mu)$. This implies that $j \mid l$. But since $(j, l) = 1$, $j = 1$ and $n \mid m$. Q.e.d.

We will now prove the theorem. Suppose we have polynomials $P$ and $Q$ of form (4) with $n \neq 0$ and $m \neq 0$. Let $n \leq m$. By Lemma 3 we know that $n \mid m$ so that in the notation of Lemma 2, $m/n = l$ and $j = 1$. Thus Lemma 2 implies that $g_m = cf_n^l$ where $c$ is a constant.

Make the following change of coordinates for $A^2$:

$$u = x, \quad x = u, \quad v = y - cx^l, \quad y = v + cu^l.$$

Applying equation (3) to this particular change of coordinates, the
given operation will be defined in \(uv\)-coordinates by

\[
(8) \quad (t, u, v) \rightarrow (P(t, u, v + cu'), Q(t, u, v + cu')) - cP'(t, u, v + cu').
\]

The coefficient of \(t^m\) in \(Q(t, u, v + cu') - cP'(t, u, v + cu')\) is \(g_m(u, v + cu') - cf_m(u, v + cu')\), which is zero.

Hence (8) reduces to an expression of form (4) in which the degree of \(P\) in \(t\) is \(n\) while the degree of \(Q\) in \(t\) is less than \(m\). We can repeat this process until we obtain coordinates in which either \(P\) or \(Q\) will have degree 0 in \(t\).

Thus for any regular operation of \(\Omega\) on \(A^2\), there are coordinates \(u, v\) so that the operation has the form

\[
(t, u, v) \rightarrow (u, Q(t, u, v)).
\]

But we may consider \((t, v) \rightarrow (Q(t, u, v))\) as defining an operation on the line. Now \(Q\) must satisfy the identity

\[
Q(s + t, u, v) = Q(s, u, Q(t, u, v)).
\]

Since characteristic =0, it follows that \(Q(t, u, v)\) must be of the form \(v + tf(u)\) where \(f \in \Omega[u]\). Thus in \(uv\)-coordinates the given operation has the form

\[
(t, u, v) \rightarrow (u, v + tf(u)).
\]

This completes the proof of the theorem.

REFERENCES


Wheaton College, Wheaton, Illinois