COMPLETE SETS OF REPRESENTATIONS OF ALGEBRAS

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1. Introduction and results. A classical theorem [1, Chapter XV, Theorem IV] states:

(1) Let $G$ be a finite group and $R$ a faithful representation\(^1\) of $G$ over a field $K$. Then each irreducible representation of $G$ over $K$ is a constituent of some tensor power of $R$.

The only proof of this result known to us actually requires the additional assumption that $K$ is of characteristic 0 and involves a calculation with characters which is not very revealing (to us). In an attempt to construct a more conceptual proof we have been led to a considerably more general result.

(2) Let $A$ be an algebra over a field $K$. Assume that $A$ has a basis $B$ over $K$ such that $B \cup \{0\}$ is closed under multiplication. Finally, let $R$ be a representation of $A$ which is faithful on $B \cup \{0\}$, and for each $r = 1, 2, \ldots$ let $\otimes^r R$ be the representation of $A$ defined by $(\otimes^r R)(b) = \otimes^r R(b)$ ($b \in B$) together with linearity. Then the representations $\otimes^r R$ ($r = 1, 2, \ldots$) form a complete set of representations of $A$ (in the sense that their direct sum is faithful on $A$).

Observe that the assumptions on $B$ imply that each $\otimes^r R$ really is a representation of $A$ and that $A$ is associative, but that there is no restriction on the characteristic of $K$ or the dimension of $A$ or $R$. The transition from (2) to (1) is immediately effected by applying to the group algebra of $G$ the statement (2) and the following probably well-known result, for which a proof is sketched at the end of this paper.

(3) If $\{\tau R \mid r = 1, 2, \ldots\}$ is a complete set of representations of a finite-dimensional algebra $A$, then each irreducible representation of $A$ is a constituent of some $\tau R$.

That the finiteness assumptions cannot be dropped in (1) or (3) may be seen from the following example. Let $e(k)$ be the real $2 \times 2$ matrix obtained by replacing the 12 entry of the identity matrix by $k$, $G$ the multiplicative group of all $e(k)$, $A$ the group algebra of $G$ over the reals, $B$ the set $G$ (imbedded in $A$), and $R$ the defining representation of $G$ extended to $A$. Then no tensor power of $R$ contains the one-dimensional representation $S$ of $A$ (or $G$) defined by $S(e(k)) = \exp k$ ($k$ real).

The proof of (2) depends on the following lemma.

\(^1\)Throughout this note all representations are assumed to correspond to left modules and the 0-representation is excluded from the list of irreducible representations.
(4) If \( C \) is a set of nonzero elements of a vector space \( V \), then in the strong direct sum \( \sum_{r=1}^{\infty} \otimes^r V \) the vectors \( \sum \otimes^r c \ (c \in C) \) are linearly independent.

2. Proofs. If the conclusion of (4) does not hold, there is a minimal nonempty finite subset \( D \) of \( C \) such that there are nonzero scalars \( k(d)(d \in D) \) for which

\[
\sum_{d \in D} k(d) \otimes^r d = 0 \quad (r = 1, 2, \ldots). 
\]

Since \( D \) clearly has at least two elements, there is a linear function \( v^* \) on \( V \) which is not constant on \( D \). Replacing \( r \) by \( r+s \) in (*) , taking the tensor product with \( \otimes^s v^* \), and then contracting, we get

\[
\sum_{d \in D} (k(d) \otimes^r d)v^*(d)^s = 0 \quad (r = 1, 2, \ldots ; \ s = 0, 1, 2, \ldots).
\]

Thus if \( k_1, k_2, \ldots, k_n \) are the distinct values taken by \( v^* \) on \( D \), the value \( k_1 \) being taken on the subset \( D_1 \) of \( D \), then because the van der Monde matrix \( (k_i^s) \ (1 \leq i \leq n, 0 \leq s \leq n-1) \) is nonsingular, the equations (*) hold with \( D \) replaced by \( D_1 \), contradicting the minimal nature of \( D \). Thus (4) is established.

Under the assumptions of (2) let \( a = \sum k(b)b \ (b \in B, k(b) \in K) \) be an element of \( A \) such that \( (\otimes^r R)(a) = 0 \) for \( r = 1, 2, \ldots \). Then

\[
\sum k(b) \otimes^r R(b) = 0 \quad \text{for} \ r = 1, 2, \ldots, \text{each} \ k(b) \text{is 0 by (4), whence} \ a \text{is also 0. Thus (2) is proved.}
\]

For the proof of (3) one may assume that \( \{^r R\} \) is finite and consists of finite-dimensional representations. Let \( ^r M = ^r M_0 \supseteq ^r M_1 \supseteq \ldots \) be a composition series for the \( A \)-module \( ^r M \) corresponding to \( ^r R \), and let \( N \) be an arbitrary irreducible \( A \)-module. If \( A^0 \) is the radical of \( A \), then \( A/A^0 \) is a sum of minimal left ideals. Hence there is a minimal left ideal \( I/A^0 \) such that \( IN \neq 0 \), and then there is a corresponding pair \( (r, i) \) such that \( I(^r M_i/^r M_{i+1}) \neq 0 \), since otherwise \( I \) would be nilpotent because \( \{^r R\} \) is complete and thus would be contained in \( A^0 \). If \( m \) and \( n \) are nonzero elements of \( ^r M_i/^r M_{i+1} \) and \( N \) respectively, it is then readily verified that the map \( im \rightarrow in \ (i \in I) \) is an \( A \)-module isomorphism of \( ^r M_i/^r M_{i+1} \) on \( N \). Hence (3).

Reference


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