CONCERNING NONNEGATIVE VALUED INTERVAL FUNCTIONS

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1. Introduction. In this paper we prove that if \( H \) is a real non-negative valued function of subintervals of the number interval \([a, b]\), then the integral (§2)

\[
\int_{[a, b]} H(I)
\]

exists if and only if for each real valued nondecreasing function \( m \) on \([a, b]\) the integral

\[
\int_{[a, b]} [H(I)dm]^{1/2}
\]

exists.

2. Preliminary definitions and theorems. Throughout this paper all integrals considered will be Hellinger [1] type limits of the appropriate sums (the definitions, theorems and proofs of this paper can be extended to "many valued" functions). Thus, if \( K \) is a real valued function of subintervals of the number interval \([a, b]\), the existence of \( \int_{[a, b]} K(I) \) is necessary and sufficient for the existence of \( \int_I K(J) \) for each subinterval \( I \) of \([a, b]\). Furthermore, the function \( F \) of subintervals of \([a, b]\), defined by \( F(I) = \int_I K(J) \), is additive.

If \( K \) is a real valued function of subintervals of \([a, b]\), then the statement that \( K \) is \( \Sigma \)-bounded on \([a, b]\) means that there is a subdivision \( D \) of \([a, b]\) such that the set of sums \( \sum_E K(I) \), where \( E \) is a refinement of \( D \) and the sum is taken over all intervals \( I \) of \( E \), is bounded. This implies that if \( I \) is an interval in a refinement of \( D \), then the set of sums \( \sum_Q K(J) \), where \( Q \) is a subdivision of \( I \) and the sum is taken over all intervals \( J \) of \( Q \), has a least upper bound \( L(I) \) and a greatest lower bound \( G(I) \). We now see that if each of \( R \) and \( T \) is a refinement of \( D \), and \( S \) is a refinement of each of \( R \) and \( T \), then

\[
\sum_R G(I) \leq \sum_S G(I) \leq \sum_S K(I) \leq \sum_T L(I) \leq \sum_T L(I).
\]

From this it follows that each of \( \int_{[a, b]} G(I) \) and \( \int_{[a, b]} L(I) \) exists, that \( \int_{[a, b]} G(I) \leq \int_{[a, b]} L(I) \), and that \( \int_{[a, b]} K(I) \) exists if and only if \( \int_{[a, b]} G(I) = \int_{[a, b]} L(I) \).

We state without proof a theorem of Kolmogoroff [2].

**Theorem.** If \( K \) is a real valued function of subintervals of \([a, b]\),
concerning nonnegative valued interval functions

integrable on \([a, b]\), then \(\int_{[a,b]} |K(I) - f_I K(J)| = 0\).

From this we have the following theorem.

**Theorem 1.** If \(K\) is a real valued function of subintervals of \([a, b]\)
integrable on \([a, b]\) and \(c\) is a positive number, then there is a subdivision
\(E\) of \([a, b]\) such that if \(F\) is a refinement of \(E\) and, for each interval \(I\)
of \(F\), \(M(I)\) is either \(K(I)\) or \(f_I K(J)\), then
\[
\sum_I M(I) - \int_{[a,b]} K(I) = 0.
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\[
\sum_I M(I) - \int_{[a,b]} K(I) = 0.
\]

**Proof.** By Kolmogoroff's theorem there is a subdivision \(E\) of 
\([a, b]\) such that if \(F\) is a refinement of \(E\), then
\[
\sum_I K(I) - \int_{[a,b]} f_I K(J) < c,
\]
so that if for each \(I\) in \(F\), \(M(I)\) is either \(K(I)\) or \(f_I K(J)\), then
\[
\sum_I M(I) - \int_{[a,b]} K(I) = 0.
\]

3. Two theorems about real nonnegative valued interval functions.

We now use Theorem 1 to prove an elementary theorem about an
integral of the form
\[
\int_{[a,b]} \left[ K(I) \int_I K(J) \right]^{1/2}.
\]

**Theorem 2.** If \(K\) is a real nonnegative valued function of subintervals
of \([a, b]\), integrable on \([a, b]\), then
\[
\int_{[a,b]} \left[ K(I) \int_I K(J) \right]^{1/2} = \int_{[a,b]} K(I).
\]

**Proof.** If \(I\) is a subinterval of \([a,b]\), then either \(K(I)\)
\[
\leq \left[ K(I) \int_I K(J) \right]^{1/2} \leq \int_I K(J) \] or \(\int_I K(J) \leq \left[ K(I) \int_I K(J) \right]^{1/2} \leq K(I),
\]
so that for each subdivision \(D\) of \([a, b]\) there is a sum \(\sum_D N(I)\) and a
sum \(\sum_D M(I)\) such that for each \(I\) in \(D\), each of \(N(I)\) and \(M(I)\) is
either \(K(I)\) or \(\int_I K(J)\) and \(\sum_D N(I) \leq \sum_D \left[ K(I) \int_I K(J) \right]^{1/2} \leq \sum_D M(I)\). Therefore, by Theorem 1, \(\int_{[a,b]} \left[ K(I) \int_I K(J) \right]^{1/2}\) exists
and is \(\int_{[a,b]} K(I)\).

**Theorem 3.** If each of \(H\) and \(K\) is a real nonnegative valued function
of subintervals of \([a, b]\), integrable on \([a, b]\), then
\[
\int_{[a,b]} \left[ (H(I) K(I))^{1/2} = \int_{[a,b]} \left[ \left\{ \int_I H(J) \right\} \left\{ \int_I K(J) \right\} \right]^{1/2}.
\]

**Proof.** We first see that if each of \(A, B, C, D\) is a nonnegative
number, then
\[(A + B)(C + D)\] \(\equiv (AC + BD)^{1/2}.\)

This implies that if \( R \) is a refinement of the subdivision \( S \) of \([a, b]\), then
\[
\sum_R \left( \left\{ \int_I H(J) \right\} \left\{ \int_I K(J) \right\} \right)^{1/2} \leq \sum_S \left( \left\{ \int_I H(J) \right\} \left\{ \int_I K(J) \right\} \right)^{1/2},
\]
so that the greatest lower bound of all sums
\[
\sum_T \left( \left\{ \int_I H(J) \right\} \left\{ \int_I K(J) \right\} \right)^{1/2}
\]
for all subdivisions \( T \) of \([a, b]\), is \( \int_{[a, b]} \left[ \left\{ \int_I H(J) \right\} \left\{ \int_I K(J) \right\} \right]^{1/2}. \) If \( E \) is a subdivision of \([a, b]\), then
\[
\left| \sum_E \left( \left\{ \int_I H(J) \right\} \left\{ \int_I K(J) \right\} \right)^{1/2} \right| = \left| \sum_E \left( \left\{ H(I) \right\} \left\{ K(I) \right\} \right)^{1/2} \right|
\]
\[
= \left| \sum_E \left( \left\{ H(I) \right\} \left\{ K(I) \right\} \right)^{1/2} \right|
\]
\[
= \left| \sum_E \left( \left\{ H(I) \right\} \left\{ K(I) \right\} \right)^{1/2} \right|
\]
\[
\leq \left\{ \sum_E H(I) \right\} \left\{ \sum_E K(I) \right\} \cdot \left\{ \int_I H(J) \right\} \left\{ \int_I H(J) \right\} \cdot \left\{ \int_I K(J) \right\} \right|^2,
\]
so that by Theorem 2, \( \int_{[a, b]} [H(I)K(I)]^{1/2} \) exists and is
\[
\int_{[a, b]} \left[ \left\{ \int_I H(J) \right\} \left\{ \int_I K(J) \right\} \right]^{1/2}.
\]

4. The existence theorem. We now establish the main result of this paper.

Theorem 4. If \( H \) is a real nonnegative valued function of subintervals of \([a, b]\), then the following two statements are equivalent:

1. If \( m \) is a real valued nondecreasing function on \([a, b]\), then \( \int_{[a, b]} [H(I)dm]^{1/2} \) exists, and
2. \( \int_{[a, b]} H(I) \) exists.

Proof. The fact that (1) follows from (2) is an immediate consequence of Theorem 3.

Suppose (1) is true. We first show that if \( x \) is in \([a, b]\), then there is a positive number \( d \) and a number \( M \) such that if \( y \) is in \([a, b]\) and \( x - d < y < x \), then \( H[y, x] \leq M \), and if \( x < y < x + d \), then \( H[x, y] \leq M \). For suppose \( a < y \leq b \) and there is an increasing sequence \( s_1, s_2, s_3, \ldots \) of numbers of \([a, b]\) such that \( 0 < H[s_n, y] \leq H[s_{n+1}, y] \) for each positive integer \( n \), and \( s_n \to y \) and \( H[s_n, y] \to \infty \) as \( n \to \infty \). There is a non-
decreasing function \( m \) on \([a, b]\) such that \( m(y) - m(z_n) = H[z_n, y]^{-1/2} \) for each positive integer \( n \), so that \( H[z_n, y]^{-1/2} \) goes to zero as \( n \to \infty \). This implies that \( \int_{[a, b]} [H(I)dm]^{1/2} \) does not exist, a contradiction. A similar argument holds for \([y, b]\) if \( a \leq y < b \).

We now show that \( H \) is \( \Sigma \)-bounded on \([a, b]\). Suppose that this is not true. The greatest lower bound \( t \) of the set of all numbers \( x \), such that \( a < x \leq b \) and \( H \) is not \( \Sigma \)-bounded on \([a, x]\), is such that either (1) \( a < t \leq b \) and if \( a \leq y < t \), then \( H \) is not \( \Sigma \)-bounded on \([y, t]\); or (2) \( a \leq t < b \) and if \( t < y \leq b \), then \( H \) is not \( \Sigma \)-bounded on \([t, y]\). We assume the former of these cases. A contradiction follows from the latter in a similar manner. This and the preceding paragraph imply that there is an increasing sequence \( z_1, z_2, z_3, \ldots \) of numbers of \([a, b]\) such that \( z_n \to t \) and \( \sum_{k=1}^{n} H[z_k, z_{k+1}] \to \infty \) as \( n \to \infty \). It follows that there is a sequence \( b_1, b_2, b_3, \ldots \) of nonnegative numbers such that \( \sum_{k=1}^{n} b_k \) converges and \( \sum_{k=1}^{n} (H[z_k, z_{k+1}] b_k)^{1/2} \to \infty \) as \( n \to \infty \). There is a nondecreasing function \( m \) on \([a, b]\) such that for each positive integer \( n \), \( m(z_{n+1}) - m(z_n) = b_n \). Since \( z_n \to t \) and \( \sum_{k=1}^{n} (H[z_k, z_{k+1}] \{m(z_{k+1}) - m(z_k)\})^{1/2} \to \infty \) as \( n \to \infty \), it follows that \( \int_{[a, b]} [H(I)dm]^{1/2} \) does not exist, a contradiction.

Therefore there is a subdivision \( D \) of \([a, b]\) and a number \( M \) such that if \( E \) is a refinement of \( D \), then \( \sum_E H(I) \leq M \). For each interval \( I \) of a refinement of \( D \), the set of sums \( \sum_Q K(J) \), where \( Q \) is a subdivision of \( I \), has a least upper bound \( L(I) \) and a greatest lower bound \( G(I) \). We let each of \( l \) and \( g \) denote a function on \([a, b]\) such that \( l(a) = g(a) = 0 \), and for \( a < x \leq b \), \( l(x) = \int_{[a, x]} L(J) \) and \( g(x) = \int_{[a, x]} G(J) \). We see that each of \( l \) and \( g \) is a real valued nondecreasing function on \([a, b]\).

Suppose \( m \) is a nondecreasing function on \([a, b]\) and \( c \) is a positive number. There is a subdivision \( A \) of \([a, b]\) such that if each of \( E \) and \( E' \) is a refinement of \( A \), then \( \left| \sum_E [H(I)dm]^{1/2} - \sum_{E'} [H(I)dm]^{1/2} \right| < c/16 \). By Theorem 3 there is a refinement \( B \) of \( D \) such that if each of \( F \) and \( F' \) is a refinement of \( B \), then

\[
\left| \int_{[a, b]} [dldm]^{1/2} - \sum_F [L(I)\Delta m]^{1/2} \right| < c/8
\]

and

\[
\left| \int_{[a, b]} [dgm]^{1/2} - \sum_{F'} [G(I)\Delta m]^{1/2} \right| < c/8.
\]

There is a common refinement \( D' \) of \( A \) and \( B \). For each interval \( I \) in \( D' \) there is a subdivision \( R_I \) and a subdivision \( S_I \) of \( I \) such that \( 0 \leq L(I) - \sum_{R_I} H(J) < c^2/16N \) and \( 0 \leq \sum_{S_I} H(J) - G(I) < c^2/16N \), where \( N \) is the number of intervals in \( D' \), so that
\[ 0 \leq \sum_{R_j} [L(J) - H(J)] < c^2/16N \]

and

\[ 0 \leq \sum_{S_j} [H(J) - G(J)] < c^2/16N. \]

It follows that

\[ 0 \leq \sum_{D'} \sum_{R_j} \left\{ [L(J)]^{1/2} - [H(J)]^{1/2} \right\} \Delta m \right\}^{1/2} \]

\[ \leq \left\{ \sum_{D'} \sum_{R_j} L(J) - 2[L(J)H(J)]^{1/2} + H(J) \right\}^{1/2} \left\{ \sum_{D'} \sum_{R_j} \Delta m \right\}^{1/2} \]

\[ \leq \left\{ \sum_{D'} \sum_{R_j} L(J) - 2H(J) + H(J) \right\}^{1/2} \left\{ \Delta m - m(a) \right\}^{1/2} \]

\[ \leq (c/4) \{m(b) - m(a)\}^{1/2}, \]

and

\[ 0 \leq \sum_{D'} \sum_{S_j} \left\{ [H(J)]^{1/2} - [G(J)]^{1/2} \right\} \Delta m \right\}^{1/2} \]

\[ \leq \left\{ \sum_{D'} \sum_{S_j} H(J) - 2[H(J)G(J)]^{1/2} + G(J) \right\}^{1/2} \left\{ \sum_{D'} \sum_{S_j} \Delta m \right\}^{1/2} \]

\[ \leq \left\{ \sum_{D'} \sum_{S_j} H(J) - 2G(J) + G(J) \right\}^{1/2} \left\{ \Delta m - m(a) \right\}^{1/2} \]

\[ \leq (c/4) \{m(b) - m(a)\}^{1/2}. \]

It therefore follows that

\[ |\int_{[a,b]} [ddm]^{1/2} - \int_{[a,b]} [dgdm]^{1/2}| \leq c/8 + (c/4) \{m(b) - m(a)\}^{1/2} + c/16 + (c/4) \{m(b) - m(a)\}^{1/2} + c/8. \]

Therefore \( \int_{[a,b]} [ddm]^{1/2} = \int_{[a,b]} [dgdm]^{1/2}. \)

If \( U \) is a subdivision of \([a,b]\), then

\[ l(b) - g(b) = \sum_{\Delta l} \Delta l - \sum_{\Delta g} \Delta g = (\sum_{\Delta l} [\Delta l]^{1/2} - \sum_{\Delta g} [\Delta g]^{1/2}) + (\sum_{\Delta l} [\Delta l]^{1/2} - \sum_{\Delta g} [\Delta g]^{1/2}), \]

so that

\[ l(b) - g(b) = \left( \int_{[a,b]} [ddl]^{1/2} - \int_{[a,b]} [dgd]^{1/2} \right) \]

\[ + \left( \int_{[a,b]} [dld]^{1/2} - \int_{[a,b]} [dgd]^{1/2} \right) = 0. \]

Therefore \( \int_{[a,b]} H(T) \) exists.

**References**


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