NON-AMPHICHEIRALITY OF THE SPECIAL ALTERNATING LINKS

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Introduction. An oriented (polygonal) link[2] k is said to be *amphicheiral* if it is isotopic to its mirror imaged link. Our study started in the direction of proving the following conjecture.

CONJECTURE. *All special alternating[3] links are not amphicheiral.*

As is well known, all torus knots are not amphicheiral. Any torus link that is alternating and prime is an elementary torus link, i.e. a torus link of type $(n, 2)$, and any prime special alternating link for which $\Delta(0)=1$ is an elementary torus link, where $\Delta(t)$ denotes the A-polynomial[4] of a link. In this paper, the following will be proved.

THEOREM A. *A special alternating link for which* $\Delta(0)=2$ *is not amphicheiral.*

THEOREM B. *A special alternating knot of odd genus is not amphicheiral.*

1. Let κ be a link type and let k be an oriented link representing κ. Let K be a (*normed regular*) projection of k onto $S^2 \subset S^3$ [8]. K is oriented by the orientation induced by that of k. Let $N(K)$ denote the number of crossing points in K and let $N(\kappa)$ denote the minimum of the crossing points in projections of any link representing κ. $N(\kappa)$ is a well-known link invariant. A projection K is said to be *reduced* if $N(K)=N(\kappa)$. Then we have the following classical theorem [2; 4].

THEOREM 1.1. *If a reduced projection of k is special alternating, then k is not amphicheiral.*

Although every link possesses at least one special projection, there exist alternating links that have no special alternating projection [5]. In the following by the *projection* of an alternating, or special alternating, link will be meant an *alternating*, or *special alternating*, projection.

Since the product of the second torsion numbers of a knot is equal

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[2] A link of multiplicity $\mu(\geqq 1)$ is the union of μ ordered, pairwise disjoint simple closed curves in 3-sphere S^3. A knot is a link of multiplicity one.

[3] For the definition, see [5, p. 278].

[4] For the definition, see [5, p. 280 and (1.11)]. We may assume without loss of generality that the leading coefficient of $\Delta(t)$, i.e. $\Delta(0)$, is greater than zero.

to $|\Delta(-1)|$, Goeritz's Theorem [8, p. 30] simply implies the following

THEOREM 1.2. *If* $|\Delta(-1)| \equiv 3 \pmod 4$ *then the knot is not amphicheiral.*

2. Theorem B will be shown from Theorem 1.2 and the following

LEMMA 2.1. *If k is a special alternating knot of genus m, then* $|\Delta(-1)| \equiv 2m+1 \pmod 4$.

PROOF. The A-polynomial of k is of the form [1; 5]:

$$\Delta(t) = a_0 - a_1 t + a_2 t^2 + \cdots + (-1)^i a_i t^i + \cdots + a_{2m} t^{2m},$$

where $a_0, a_1, \cdots, a_{2m} > 0$. Then we have $\Delta(-1) = \sum_{i=0}^{2m} a_i > 0$. Since $a_j = a_{2m-j}$, for $j = 0, 1, \cdots, m$, it follows that $\Delta(-1) = 2(a_0 + a_1 + \cdots + a_{m-1}) + a_m$. On the other hand, since $\Delta(1) = \sum_{i=0}^{2m} (-1)^i a_i = 1$ [5], we have

$$\begin{aligned} \Delta(-1) &= 2(a_0 + a_1 + \cdots + a_{m-1}) \\ &\quad + (-1)^m \{1 - 2[a_0 - a_1 + \cdots + (-1)^{m-1} a_{m-1}]\} \\ &= 2\{[1 + (-1)^{m+1}] a_0 + [1 + (-1)^{m+2}] a_1 + \cdots \\ &\quad + [1 + (-1)^{2m}] a_{m-1}\} + (-1)^m \\ &\equiv (-1)^m \equiv 1 + 2m \pmod 4. \end{aligned}$$

3. Let G be a (*finite planar*), *directed* or *undirected*, *graph*. A sequence $P = (a_0, A_1, a_1, \cdots, A_n, a_n)$, a_i and A_j denoting vertices and edges, is an *undirected path* in G from a_0 to a_n if $a_0, a_1, \cdots, a_{n-1}$ are all distinct and if a_{j-1} and a_j are two ends in G of A_j for $j = 1, 2, \cdots, n$. Specially a path P is said to be *directed* from a_0 to a_n if A_j are directed from a_{j-1} to a_j. A path is called a *circuit* (or *n-circuit*) if $a_0 = a_n$. The *valency* of a vertex is defined as the number of edges in G incident to it.

Now let K be a projection of a link k. Then we can uniquely define a graph $G(K)$ and a dual graph $G^*(K)$ of K [5]. To each crossing in K there correspond a unique edge of $G(K)$ and a unique edge of $G^*(K)$. Each edge of $G(K)$ is directed in such a way that it crosses the overpassing segment from left to right at the crossing point corresponding to it. Thus $G(K)$ and $G^*(K)$ are directed. If k is a special alternating link, then we can choose the graph $G(K)$ in such a way that the valency of each vertex of $G(K)$ is even and $G(K)$ is the union of directed circuits.

Conversely given an undirected graph G, there are exactly two *alternating projections*. One is the projection of the mirror imaged link of the other. If they are oriented, then G is directed. Since the A-polynomial of a mirror imaged link coincides with that of an original link, the A-polynomial of the alternating link induced from a given graph G is uniquely determined. Thus we can call it the *A-polynomial* of G.

We can assume, hereafter, that $G(K)$ and $G^*(K)$ are connected and contain no 1-circuit.

A projection K is of the *bridge form of type* $(p_1, p_2, \cdots, p_r)$ if $G(K)$ or $G^*(K)$ consists of r paths $P_1, P_2, \cdots, P_r$ from a vertex u to another vertex v, having no common vertices except u and v, where P_i contain p_i vertices other than u and v. k is called, then, a *bridge link of type* $(p_1, p_2, \cdots, p_r)$. If k is a special alternating bridge link of type $(p_1, p_2, \cdots, p_r)$, then in $G(K)$, r is always even, and one half of r paths are directed from u to v and the other half are directed reversely. The number of vertices of $G(K)$ equals[5] $d(K)+1$. For example, the special alternating bridge link of type (p, q) coincides with the torus link of type $(p+q+2, 2)$. Moreover, a special alternating knot of genus one is an alternating bridge knot of type (p, q, r).

4. To characterize the links for which $\Delta(0)=2$, we need the following

LEMMA 4.1. *Let G be the directed graph of a projection K of a special alternating link and let G' be the directed graph obtained from G by removing a directed circuit C. Denoting the A-polynomials of G and G' by $\Delta(t)$ and $\Delta'(t)$ respectively, we have*

$$\Delta(0) \geqq \Delta'(0) + 2^{n-1} - 1, \tag{4.1}$$

where n denotes the number of vertices in $C\cap G'$.

PROOF. Let us denote $I_{p,q}(i, j)$ the $p\times q$ matrix whose elements are all zero except the (i, j) element, which is equal to one. By definition, then, knot matrices[6] M, M' of G, G' are of the forms:

$$M = \begin{bmatrix} A_{00} & A_{01} & \cdots A_{0,n+1} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ A_{n+1,0} & A_{n+1,1} & \cdots A_{n+1,n+1} \end{bmatrix}, \qquad M' = \begin{pmatrix} A & A_{0,n+1} \\ A_{n+1,0} & A_{n+1,n+1} \end{pmatrix},$$

where $A_{0i} = -I_{n,p_i}(i, 1)$, $A_{j,0} = -I_{p_j,n}(p_j, j+1)$, $A_{n,0} = -I_{p_n,n}(p_n, 1)$,

[5] $d(k)$ or $d(K)$ denotes the degree of the A-polynomial $\Delta(t)$ of k.

[6] By the *knot matrix of G* is meant the knot matrix of the alternating link induced by G. For the definition and properties of a knot matrix, and for the matrix notations, see [5; 6].

$$A_{kk} = I - \sum_{i=1}^{p_k-1} I_{p_k,p_k}(i, i+1),$$

$$(i = 1, \cdots, n, j = 1, \cdots, n-1, k = 1, \cdots, n, p_l \geqq 0 \text{ for all } l),$$

$A = A_{00} - I$, I denoting the identity matrix, and $A_{ij} = 0$ $(i, j=1, \cdots, n+1, i\neq j)$. A_{00} and A correspond to n common vertices in $C\cap G'$. Since $\Delta(0) = \det \tilde{M}(n, n)$ and $\Delta'(0) = \det \tilde{M}'(n, n)$ by definition, it follows easily that

$$\Delta(0) = \Delta'(0) + \sum_{\substack{1\leq i_1<i_2<\cdots<i_\lambda\leq n-1 \\ \lambda=1,2,\cdots,n-1}} \det N_{i_1 i_2\cdots i_\lambda};$$

$N_{i_1 i_2\cdots i_\lambda}$ is the matrix obtained from $\tilde{M}'(n, n)$ by replacing the i_j-row $(1\leqq j\leqq\lambda)$ by $R_{i_j} = I_{1,n+s-1}(1, i_j) - I_{1,n+s-1}(1, i_j+1)$, where s denotes the number of columns of $A_{0,n+1}$ and we define $R_{n-1} = I_{1,n+s-1}(1, n-1)$. To prove (4.1) it is sufficient to show that $\det N_{i_1 i_2\cdots i_\lambda} > 0$. This is obvious from Lemma 2.4 (i) in [**5**], since $N_{i_1\cdots i_\lambda}$ is of strongly special type on the row.[7] q.e.d.

If $n=2$, the Lemma 4.1 is sharpened as follows.

LEMMA 4.2. $\Delta(0) = \Delta'(0) + \Delta''(0)$, *where* $\Delta''(t)$ *denotes the* A*-polynomial of the graph obtained from* G' *by identifying two common vertices.*

By using Lemma 4.1, we can complete the characterization. In fact, we have

LEMMA 4.3. *A prime special alternating link* k *for which* $\Delta(0)=2$ *is a special alternating bridge link of type* (p, q, r, s), *where* $p+q+r+s+1 = d(k)$, *and conversely.*

PROOF. Let K be a projection of k and let G' be the graph obtained in Lemma 4.1. Then (4.1) shows that $2=\Delta(0)\geqq\Delta'(0)+2^{n-1}-1$. This implies that $\Delta'(0)=1$ and $n=2$, since k is prime. Thus K is of the form as is required in the lemma. It is obvious that $p+q+r+s+1 = d(k)$. Converse is clear.

5. To prove Theorem A, some lemmas on $N(\kappa)$ will be required. In Lemmas 5.1–5.3, k need not be alternating.

LEMMA 5.1. *For any nontrivial link* k, $N(\kappa)\geqq d(k)+1$. *Equality holds if and only if* k *is an elementary torus link.*

This is obvious from [**1**; **5**].

LEMMA 5.2. *For any special projection* K *of any link* k *for which* $\Delta(0)=2$, $N(K)\geqq d(k)+3$.

[7] For the definition, see [5, p. 289].

PROOF. From Lemma 5.1, we see that $G(K)$ contains at least $d(k)+1$ edges. Since K is special, the valencies of all vertices must be even. If the valency of each vertex is two, then G is a circuit, since G is connected. Thus we have $\Delta(0)=1$, which is a contradiction. If only one vertex is of valency four, then G consists of two circuits having one vertex in common, from which it follows that $\Delta(0)=1$. Thus G contains at least two vertices of valency $\geqq 4$. It follows that $N(K)\geqq d(K)+3$.

LEMMA 5.3. *For any nonspecial projection K of any link*, $N(K) > d(K)+3$.

PROOF. We can define a matrix for any link [5; 6]. Then Corollary 1.40 in [5] can be extended as follows. Let $K=K_1 * K_2 * \cdots * K_p$ $(p\geqq 2)$,[8] and let $\Delta_i(t)$ denote the A-polynomials of K_i. Then we have

$$(5.1) \qquad \Delta(0) = \prod_{i=1}^{p} \Delta_i(0) \quad \text{and} \quad d(K) = \sum_{i=1}^{p} d(K_i).$$

Thus all $\Delta_i(0)$ except one, say $\Delta_p(0)$, are equal to one, and $\Delta_p(0)=2$. Hence from Lemmas 5.1, 5.2 we have

$$(5.2) \qquad N(K_i) \geqq d(K_i) + 1 \quad \text{and} \quad N(K_p) \geqq d(K_p) + 3.$$

Consequently it follows from (5.1) and Lemma 3.29 in [5],

$$\begin{aligned} N(K) &= \sum_{i=1}^{p-1} (d(K_i) + 1) + d(K_p) + 3 \\ &= \sum_{i=1}^{p} d(K_i) + (p-1) + 3 \\ &= d(K) + p + 2 > d(K) + 3, \end{aligned}$$

q.e.d.

Now any special alternating bridge link k of type (p, q, r, s) possesses a projection K with $d(K)+3$ crossing points, which is reduced and special alternating. Thus from Theorem 1.1 it follows that k is not amphicheiral. Combining with Lemma 4.2, we have Theorem A.

APPENDIX. Since a special alternating bridge link of type $(0, 0, p, q)$ $(p, q\geqq 0)$ is of two bridges, we can prove the non-amphicheirality of links of this kind by means of a theorem of H. Schubert [9]. In fact, we see that $\alpha=2pq+3p+3q+4$ and $\beta=2p+3$ or $2q+3$, where (α, β) denotes the normal form of this link [7; 9]. From this it is easy to show that $\beta^2\not\equiv -1 \pmod{2\alpha}$.

[8] For the definition, see [5, p. 293].

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