ISOMORPHISMS OF THE ENDOMORPHISM RINGS OF TORSION-FREE MODULES

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Let (R, M) denote the unitary left module M over the ring R, and let $E(R, M)$ denote the ring of all R endomorphisms of M. The purpose of this note is to show that the technique used by Kaplansky in [**3**] can be used to prove the following:

THEOREM. *For $i=1, 2$ let M_i be a torsion-free module over the complete discrete valuation ring R_i, and suppose α is an isomorphism of $E(R_1, M_1)$ upon $E(R_2, M_2)$. Then the modules M_1 and M_2 are either both nondivisible or both divisible. In the first case, there exists a one-one semilinear mapping V of (R_1, M_1) onto (R_2, M_2) such that $S^\alpha = V^{-1}SV$ for each S in $E(R_1, M_1)$. In the latter case, M_i is a vector space over K_i (the quotient field of R_i) and α is induced by a one-one semilinear transformation of (K_1, M_1) upon (K_2, M_2).*

LEMMA 1. *Let R be a complete discrete valuation ring, and M a torsion-free R module which is not divisible. Then the center of $E(R, M)$ is R.*

PROOF. By Theorem 3, and Corollary 1 to Theorem 23 of [**3**], M has a direct summand isomorphic to R. The desired result now follows by [**3**, p. 73, problem 95].

LEMMA 2. *Let R be an integral domain with quotient field K, and let M be a torsion-free divisible R module. Then $E(R, M) = E(K, M)$.*

PROOF. By [**3**, Theorem 4] M is a vector space over K. Now let $\sigma \in E(R, M)$ and $k \in K$. Then $k = r/s$, $r, s \in R$, $s \neq 0$.

Then $(kx)\sigma = (r/s\cdot x)\sigma = (r\cdot 1/s\cdot x)\sigma = r(1/s\cdot x)\sigma = r/s\cdot s[(1/s\cdot x)\sigma] = r/s\cdot[s\cdot(1/s\cdot x)]\sigma = r/s(x\sigma) = k(x\sigma)$ and $\sigma \in E(K, M)$ completing the proof.

LEMMA 3. *Assume the hypotheses of the Theorem, and in addition that R_1 is not a field, and M_1 is cyclic. Then M_2 is cyclic, hence isomorphic to R_2.*

PROOF. The ring of endomorphisms $E(R_1, M_1)$ is isomorphic to R_1 [**2**, p. 80]. Hence $E(R_2, M_2)$ is isomorphic to R_1, and hence is an integral domain. But then M_2 must be indecomposable. For suppose

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$M_2 = P \oplus Q$, with P, Q proper submodules. Let e be the projection on P which annihilates Q; then $e(1-e)=0$ where 1 is the identity endomorphism, showing that $E(R_2, M_2)$ has proper zero divisors. By Corollary 2 to Theorem 23 of [3], M_2 is either isomorphic to R_2 or to K, the quotient field of R_2. If M_2 is isomorphic to K, $E(R_2, M_2) = E(R_2, K) = E(K, K)$ (by Lemma 2), hence is isomorphic to K. But then R_1 is isomorphic to K, and hence is a field, contradicting the hypothesis. Thus M_2 is cyclic and isomorphic to R_2.

Proof of theorem. We consider first the case in which neither M_1 nor M_2 is divisible. Then neither R_1 nor R_2 can be fields, since vector spaces over fields are divisible modules.

By Theorem 3 and Corollary 1 to Theorem 23 of [3], $M_1 = R_1 x \oplus Q$ with $x \in M_1$. Let e be the projection on $R_1 x$ which annihilates Q, and put $e^\alpha = f$. By [3, p. 71, exercise 84] and Lemma 3, $M_2 f$ is a cyclic module over R_2. Let $M_2 f = R_2 y$ with $y \in M_2$. Let z be arbitrary in M_1. Then there is defined a unique endomorphism $C(z) \in E(R_1, M_1)$ by the conditions

$$xC(z) = z,$$
$$QC(z) = 0.$$

Now define the single-valued mapping V of M_1 into M_2 by $zV = y[C(z)^\alpha]$.

Since the isomorphism α maps the center of $E(R_1, M_1)$ onto the center of $E(R_2, M_2)$ it follows from Lemma 1 that R_1 and R_2 are isomorphic. Let σ_r denote the endomorphism $x \to rx$ where $x \in M_i$, $r \in R_i$ ($i = 1$ or 2), and suppose $(\sigma_r)^\alpha = \sigma_s$, $r \in R_1$, $s \in R_2$. Then

$$x[C(z)\sigma_r] = [xC(z)]\sigma_r = rz,$$
$$Q[C(z)\sigma_r] = 0.$$

Hence $(rz)V = y[C(z)\sigma_r]^\alpha = yC(z)^\alpha(\sigma_r)^\alpha = (zV)\sigma_s = s(zV)$ or $(rz)V = r^\alpha \cdot (zV)$ where $r \to r^\alpha$ is the isomorphism of R_1 onto R_2 induced by the isomorphism of $E(R_1, M_1)$ onto $E(R_2, M_2)$.

We can now proceed as in [3] to show that V is a one-one semilinear mapping of (R_1, M_1) onto (R_2, M_2) that induces the ring isomorphism α.

Now assume it possible that M_2 is divisible but that M_1 is not. Then R_1 is not a field. We proceed as in the previous case, to conclude that M_2 has an indecomposable direct summand $M_2 f$ isomorphic to R_2. As a direct summand of a divisible module, $M_2 f$ is divisible and hence also isomorphic to K_2. Thus R_2 is isomorphic to K_2, and is thus a field. By Lemma 1, the center of $E(R_1, M_1)$ is R_1. The center of the ring of linear transformations of a vector space over a field consists of the scalar multiplications; so the center of $E(R_2, M_2)$ is R_2. Since the isomorphism α induces an isomorphism of the centers, we have R_1 and R_2 isomorphic, which is a contradiction since R_2 is a field, while R_1 is not.

Finally, suppose M_1 and M_2 are both divisible. By [**3**, Theorem 4] they are vector spaces over K_1, and K_2 respectively. By Lemma 2, $E(R_i, M_i) = E(K_i, M_i)$, $i = 1, 2$. We may therefore apply the theorem on isomorphisms of rings of linear transformations [**1**, p. 183] to conclude the existence of a semilinear transformation of (K_1, M_1) onto (K_2, M_2) which induces α.

REMARK 1. In the case in which M_1 and M_2 are divisible, there need not exist a semilinear mapping of the module (R_1, M_1) onto the module (R_2, M_2). For let R be a complete discrete valuation ring which is not a field, and let K be its quotient field. Then if we put $M_1 = M_2 = K$, and $R_1 = R$, $R_2 = K$ we have $E(R_1, M_1)$ and $E(R_2, M_2)$ isomorphic by Lemma 2. The existence of a semilinear mapping would make R and K isomorphic, a contradiction.

REMARK 2. If M is a torsion-free module over the complete discrete valuation ring R (with quotient field K) then it follows from Lemmas 1 and 2 that the center of $E(R, M)$ is either K or R depending on whether M is divisible or not. Hence any automorphism of $E(R, M)$ leaving the center elementwise fixed is inner. This latter statement is stated as an exercise in [**3**, p. 73, problem 98].

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