ISOMETRIC IMMERSIONS WHICH PRESERVE CURVATURE OPERATORS

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The curvature tensor of a Riemannian manifold M can be expressed by a function which assigns to each pair of vectors $x, y \in M_m$ (tangent space to M at m) a skew-symmetric linear operator R_{xy} on M_m [1]. Call R_{xy} the *curvature operator* of x, y. Let $j: M^d \to \overline{M}^{d+1}$ be an isometric immersion. If j is totally geodesic, then j *preserves curvature operators*, that is, if $x, y, z \in M_m$, then $dj(R_{xy}(z)) = \overline{R}_{dj(x),dj(y)}(dj(z))$. The converse is generally false. We are going to consider the character of immersions as above which preserve curvature operators. The simplest example is an arbitrary isometric immersion of R^d in R^{d+1}. In particular we show that if the domain M^d of j is complete and has positive curvature then the converse above holds, that is, if j preserves curvature operators, then j is totally geodesic.

1. **General case.** Note that $j: M^d \to \overline{M}^{d+1}$ preserves curvature operators if and only if (a) j preserves Riemannian curvature, i.e. $\overline{K}(dj(\pi)) = K(\pi)$ for all 2-planes π tangent to M, and (b) if $z \in \overline{M}_{j(m)}$ is orthogonal to $dj(M_m)$, then $\overline{R}_{dj(x),dj(y)}(z) = 0$ for all $x, y \in M_m$. The proof is elementary, and depends on the fact that the codimension of M in $\overline{M}$ is one.

THEOREM 1. *Let $j: M^d \to \overline{M}^{d+1}$ be an isometric immersion which preserves curvature operators, and let M be complete. Then the open set N of nongeodesic points of M rel. j is foliated by complete $(d-1)$-dimensional submanifolds which are totally geodesic rel. j.*

PROOF. Since j preserves Riemannian curvature, at each point of M there is at most one curvature direction with nonzero principal curvature. Thus on the set N of nongeodesic points, the directions of zero normal curvature constitute a differentiable field $\mathcal{P}$ of $(d-1)$-planes. We will integrate $\mathcal{P}$ to obtain the required foliation. (The theorem holds trivially when N is empty.)

Each point of N has a neighborhood U on which there is a unit normal vector field E_{d+1} rel. j and a frame field $E = (E_1, \cdots, E_d)$ whose first vector is in the curvature direction with principal curvature $\kappa_1 \neq 0$. From the frame field E one obtains on U the dual-base forms ω_i, the Riemannian connection forms ϕ_{ij}, and curvature forms Φ_{ij} of M, $1 \leqq i, j \leqq d$. Enlarging E by adding E_{d+1} to it, we get the Codazzi forms σ_i, $1 \leqq i \leqq d$, and curvature forms $\overline{\Phi}_{rs}$, $1 \leqq r, s \leqq d+1$,

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of $\overline{M}$. Dropping the differential map of j from the notation, we can write $\overline{R}_{E_iE_j}(E_{d+1}) = -\sum_k \overline{\Phi}_{k,d+1}(E_i, E_j)E_k$. Thus by (b) above, we have $\overline{\Phi}_{k,d+1}=0$ on U. Furthermore, $\sigma_1=\kappa_1\omega_1\neq 0$, and $\sigma_i=0$ if $i>1$. Thus the Codazzi equations $d\sigma_i = -\sum_k \phi_{ik}\wedge\sigma_k + \overline{\Phi}_{d+1,i}$ reduce to $d\sigma_1=0$ and $\phi_{i1}\wedge\sigma_1=0$. Since σ_1 annihilates the planes of $\mathcal{P}$, $d\sigma_1=0$ implies $\mathcal{P}$ is integrable. The other equations imply that the forms ϕ_{i1} are zero on vectors tangent to a leaf L of $\mathcal{P}$. But these forms, $1<i\leqq d$, are the Codazzi forms for L in M, so each leaf L is totally geodesic in M—and hence also in $\overline{M}$, i.e. rel. j.

Now we show that the leaves L are complete by showing that geodesics of L are infinitely extendible. Suppose the contrary, i.e. that there is a maximal geodesic α of a leaf L which is defined only on a bounded open interval (a, b). Since M is complete, α is infinitely extendible as a geodesic of M. Since L is totally geodesic, as long as this extension $\tilde{\alpha}$ remains in N, it is a geodesic of L. So the limit points $\tilde{\alpha}(a)$ and $\tilde{\alpha}(b)$ of α are not in N. We will contradict this by showing that κ_1 is zero at neither of these points. We can assume that the geodesic segment α (but not its limit points) lies in the domain of fields E and E_{d+1} as above, with the further properties that α is an integral curve of E_2 and that E is parallel on α. In fact, once E is properly defined on α, one can extend over a neighborhood of α in M by first extending over a neighborhood in the leaf L, keeping E_1 perpendicular to L, then extending over the full neighborhood, keeping E_1 always in the κ_1 curvature direction. (Strictly speaking, one passes to a suitable covering manifold if α crosses itself.)

From the first structural equation, we deduce $[E_1, E_2] = \sum \phi_{i2}(E_1)E_i$. Applying the form $d\sigma_1=0$ to the fields E_1, E_2 gives $E_2(\kappa_1) = -\kappa_1\phi_{12}(E_1)$. Setting $k=\kappa_1\circ\alpha$, $f=\phi_{12}(E_1)\circ\alpha$, we write this equation as

$$k' = -kf. \tag{1}$$

Applying the second structural equation to the fields E_1, E_2 and simplifying, using the facts above, we get $E_2(\phi_{12}(E_1)) = -(\phi_{12}(E_1))^2 - \Phi_{12}(E_1, E_2)$. Setting $F=\Phi_{12}(E_1, E_2)\circ\alpha$ yields

$$f' = -f^2 - F. \tag{2}$$

Our assumption that L is not complete has led to the conclusion that $k(t)$ approaches zero as t approaches either a or b. The differential equations (1) and (2) contradict this. In fact, solving (1) explicitly, we deduce that as $t\to b$, $\limsup f = +\infty$. This contradicts (2) which says, since F is bounded below on (a, b), that when f is large enough its slope is negative. The argument when $t\to a$ is similar, so the proof is complete.

A scheme similar to that above was used by Chern and Lashof in [3, Lemma 2].

THEOREM 2. *Suppose M^d $(d \geqq 2)$ is complete and has Riemannian curvature $K>0$. Then every isometric immersion $j: M^d \to \overline{M}^{d+1}$ which preserves curvature operators is totally geodesic.*

PROOF. Suppose there is a nongeodesic point, that is (in the notation of the previous proof) N is not empty. Then a geodesic α as in that proof has domain the whole real line. Thus we can arrange for the function $f=\phi_{12}(E_1) \circ \alpha$ to be defined on the whole real line, and f satisfies the differential equation (2) $f' = -f^2 - F$. But this is impossible when $K>0$, since then $F>0$.

This is not a local result—it fails if M is not required to be complete.

2. **Constant curvature case.** If $\overline{M}^{d+1}$ has constant curvature, then its curvature operators have the property that $\overline{R}_{xy}(z)=0$ if z is perpendicular to x and y. (Converse, §177 of [2].) Thus by the first remark of the previous section, if M^d and $\overline{M}^{d+1}$ have the same constant curvature, then every isometric immersion $j: M^d \to \overline{M}^{d+1}$ preserves curvature operators. We consider the character of j and M^d when $\overline{M}^{d+1}$ is specialized to be a sphere $S^{d+1}(C)$, Euclidean space R^{d+1}, or hyperbolic space $Q^{d+1}(C)$, where C is curvature of appropriate sign. From Theorem 2 we get: *if M^d is complete and has constant curvature $C>0$, then M^d can be immersed in $S^{d+1}(C)$ if and only if M^d is isometric to $S^d(C)$. Any such immersion is an imbedding onto a great d-sphere.*

In the case $C=0$, Hartman and Nirenberg [4] have proved: *a complete flat manifold M^d can be immersed in R^{d+1} if and only if M^d is isometric to either R^d or $S^1(r) \times R^{d-1}$. Any such immersion is as a cylinder in R^{d+1}.*

This can be proved by applying Theorem 1 to both $j: M^d \to R^{d+1}$ and $j \circ \pi: R^d \to R^{d+1}$, where $\pi: R^d \to M^d$ is the universal covering of M^d. The special character of disjoint, totally geodesic hypersurfaces in R^d allows us to extend the foliation of the set N in R^d to a foliation of all of R^d by parallel $(d-1)$-planes.

This general scheme fails in the negative curvature case, since disjoint, totally geodesic hypersurfaces in $Q^d(C)$ can have more complicated arrangements. One can exhibit surfaces with curvature $C<0$ in $Q^3(C)$ with arbitrary first Betti number. However the Euclidean result can be extended topologically to the negative curvature case as follows:

THEOREM 3. *Let M^d be a complete manifold with constant negative curvature C. If M^d can be isometrically immersed in $Q^{d+1}(C)$, then $H^i(M^d)=0$ for $i \geqq 2$.*

(Here H denotes Čech cohomology with arbitrary coefficients.)

PROOF. From such an immersion j we get a decomposition of M as in Theorem 1. Denote the components of N by N_α, the components of $M-N$ by F_β. Each leaf L of N is complete and totally geodesic rel. j, hence isometric to $Q^{d-1}=Q^{d-1}(C)$. The immersion j is one-to-one on components F_β also. Let $\pi: Q^d \to M^d$ be the universal covering. Then we can derive

(1) If a subset A of M can be lifted into Q^d, so can the union of those sets L and F_β which meet A.

(2) There is a number $\epsilon>0$ such that if B, C, D are disjoint totally geodesic hypersurfaces in Q^d which meet an ϵ-neighborhood, then B, C, D are linearly ordered, i.e. some one separates the other two in Q^d.

(3) Each F_β is either a totally geodesic Q^{d-1} or (if its interior is not empty) a manifold with boundary B_β, where B_β is a union of totally geodesic sets Q^{d-1}, each of which is disjoint from the closure of the others. In particular each F_β is contractible.

By a theorem of Ricci (§107, [2]) the orthogonal trajectories of the leaves of an N_α give isometries of the leaves. If N is dense in M it follows (much as in the Euclidean case) that M is diffeomorphic to either R^d or $S^1 \times R^{d-1}$. Excluding this case we have

(4) The boundary of each N_α is either a single totally geodesic Q^{d-1} or two disjoint ones, and the closure $\overline{N}_\alpha$ of N_α is contractible.

Consider the covering $\mathcal{C}$ of M by all sets N_α and F_β. This is a closed covering by homologically trivial sets. Furthermore, any intersection of three elements of $\mathcal{C}$ is empty, and the intersection of any two consists of at most two disjoint sets Q^{d-1}. Suppose $\mathcal{C}$ is locally finite, e.g. $M-N$ only a finite number of components. Then by a well-known theorem, the cohomology of M is isomorphic to the cohomology of the nerve of $\mathcal{C}$. Since this nerve has dimension 1 the result follows. If $\mathcal{C}$ is not locally finite we can alter it, retaining its essential properties, so as to get local finiteness. We omit the details of the proof. Roughly speaking, if $\mathcal{C}$ is not locally finite at a point p, then p lies in a "limit face" Q_1 of an element, say $\overline{N}_\alpha$, of $\mathcal{C}$. Choose $N_\beta \neq N_\alpha$ sufficiently near Q_1 and let Q_2 be the face of N_β nearest Q_1. Using (1) and (2) we can define G to be the union of Q_1, Q_2, and the elements of $\mathcal{C}$ *between* Q_1 and Q_2. Finally, replace these elements by G in $\mathcal{C}$. Iteration of this operation eliminates all limit faces.

In general the complexity of the decomposition of M given by

Theorem 1 is measured by the identification space M^* whose elements are the leaves of N and the components of $M-N$. If M^* is metrizable, it can be shown to have inductive dimension 1. In this case the argument above can be replaced by an application of the Vietoris mapping theorem.

BIBLIOGRAPHY

1. W. Ambrose and I. M. Singer, *A theorem on holonomy*, Trans. Amer. Math. Soc. **75** (1953), 428–443.

2. E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, Gauthier-Villars, Paris, 1951.

3. S. Chern and R. Lashof, *On the total curvature of immersed manifolds*, Amer. J. Math. **79** (1957), 306–318.

4. P. Hartman and L. Nirenberg, *On spherical image maps whose Jacobians do not change sign*, Amer. J. Math. **81** (1959), 901–920.

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ON THE EMBEDDABILITY OF THE REAL PROJECTIVE SPACES[1]

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In a paper of the same title, Massey [4] proved that if $2^{k-1}+2^{k-\rho}-1\leqq n<2^k$ then P_n cannot be differentiably embedded in R^{2^k}. By using the technique of Massey in a different way we can prove the following theorem which clearly includes Massey's.

THEOREM. *If $2^{k-1}<n<2^k$ then P_n cannot be embedded differentiably in Euclidean space of dimension 2^k.*

Besides the result of Massey, the main result in this direction is if $2^{k-1}<n<2^k$ then P_n cannot be embedded differentiably in R^{2^k-1}. Our result yields, in particular, that for P_{2^k+1}, the embedding in $R^{2^{k+1}+1}$ given by Hopf and James [1] is the best possible.

The following information from [3; 4] will be needed. Let M be a n-manifold differentiably embedded in R^{n+k+1}; and let $p: E\rightarrow M$ denote the bundle of unit normal vectors. Then there exist subalgebras $A^*(E, Z)\subset H^*(E, Z)$ and $A^*(E, Z_2)\subset H^*(E, Z_2)$ which satisfy the following conditions:

1. $A^0(E, G)=H^0(E, G)$,
2. $H^q(E, G)=A^q(E, G)+p^*(H^q(B, G))$ $(0<q<n+k)$,
3. $A^q(E, G)=0$, $q\geqq n+k$,

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