UNKNOTTING 3-SPHERES IN SIX DIMENSIONS

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Haefliger [2] has shown that a differentiable embedding of the 3-sphere S^3 in euclidean 6-dimensions E^6 can be differentiably knotted. On the other hand any piecewise linear embedding of S^n in E^k is combinatorially unknotted if $k \geqq n+3$ (see [5; 6; 7]). The case S^3 in E^6 appears to be the first occasion on which the differentiable and combinatorial theories of isotopy differ. Therefore it seemed worthwhile to give separately the proof of the combinatorial unknotting in S^3 in E^6, because the argument in this case is considerably simpler than that in the general case [7]. The proof is similar to that of unknotting S^2 in E^5 (see [5]), although it does involve one new idea, that of "severing the connectivity of the near and far sets" (without which I had conjectured the opposite in Remark 2 of [5]).

THEOREM. *Given a piecewise linear embedding of a combinatorial 3-sphere S is euclidean 6-dimensions E^6, then it is unknotted, i.e. there is a piecewise linear homeomorphism of E^6 onto itself, throwing S onto the boundary of a 4-simplex.*

PROOF. By [6, Theorem 1] it suffices to show that S is equivalent by cellular moves (we shall in fact use three moves) to the boundary of a 4-ball. The definition of a cellular move (introduced in [6, p. 351]) is as follows: If T is another 3-sphere in E^6, we say that S is equivalent to T by a cellular move across the 4-ball Q, written $S \sim T$ across Q, if the interior of Q does not meet S, T, and $\dot{Q}$ is the union of the two 3-balls $\mathrm{Cl}(S-T)$, $\mathrm{Cl}(T-S)$.

First choose a vertex V in general position relative to S (see [6, p. 357]). If the cone VS is nonsingular we are finished, because it is a 4-ball bounded by S. Otherwise there are *singular* points $x \in S$, such that the line Vx meets S again. But owing to the general position of V, there can be at most two singular points on any line through V (see [6, Lemma 4]); therefore if we have V away on one side, we can call the one nearest to V *near-singular* and the other *far-singular*. Let nS denote the set of near-singular points of S, and fS the set of far-singular points. These two sets depend upon the position of the vertex V, of course, which we have not included in the notation, but once V has been chosen we assume it to remain fixed throughout.

For dimensional reasons the sets nS, fS are 1-dimensional, and radial projection from V establishes a homeomorphism between

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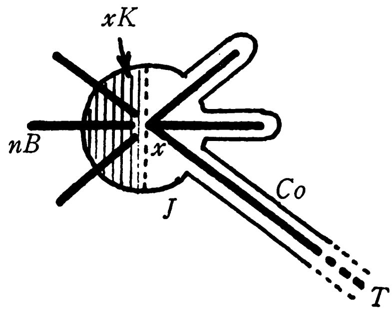
them. By construction nS, fS are disjoint, but their closures $\mathrm{Cl}(nS)$, $\mathrm{Cl}(fS)$ may intersect, in X say. If X is nonempty, it consists of a finite set of nonsingular vertices of S, and $X=\mathrm{Cl}(nS)\cap\mathrm{Cl}(fS)$ $=\mathrm{Cl}(nS)-nS=\mathrm{Cl}(fS)-fS$. Triangulate $\mathrm{Cl}(nS)$, $\mathrm{Cl}(fS)$ as isomorphic 1-complexes, so that the isomorphism is given by projection from V. (They are not subcomplexes of S, but are piecewise linearly embedded in S.)

Now we cannot proceed as in [**5**] and separate the sets nS, fS by a diameter of S, because being 1-dimensional they may link in S. Therefore the first job is to "sever the connectivity" of one of them. Let Y be the set of barycentres of the 1-simplexes of $\mathrm{Cl}(nS)$, which is a finite set of points. Let Z be a cone on Y in general position in S—that is to say Z meets $\mathrm{Cl}(nS)$ in Y, and does not meet $\mathrm{Cl}(fS)$. Let S_1 be a subdivision of S containing subcomplexes triangulating $\mathrm{Cl}(nS)$, $\mathrm{Cl}(fS)$, Z. Let A be the closed simplicial neighbourhood of Z in the second derived complex S_1'' of S_1. Then A is a (combinatorial) 3-ball by Whitehead's regular neighbourhood theorem [**4**, Theorem 23, Corollary 1], because Z is collapsible. Let $B=\mathrm{Cl}(S_1''-A)$, which is also a 3-ball by [**1**, Theorem 14: 2]. Since $\mathrm{Cl}(fS)$ does not meet Z, neither does it meet A, and so it is contained in the interior of B.

The cone VA is a 4-ball, whose interior does not meet S, because A contains only near- and non-singular points. The boundary $(VA)^\cdot = V\dot{A}+A = V\dot{B}+A$. Therefore $S=A+B$ is equivalent to the 3-sphere $V\dot{B}+B$ by a cellular move across VA.

The first stage of the proof is complete, because we have severed the connectivity of the singular sets in the following sense: Let nB, fB denote the near- and far-singular sets of the ball B, defined with respect to the fixed vertex V in the same way that nS, fS were defined for S. Since $B\subset S$, then $nB\subset nS\cap B$. Conversely, if $x\in nS\cap B$, the corresponding far-singular point $y\in fS\subset B$, and so $x\in nB$ and $y\in fB$. Hence $nB=nS\cap B$, which is nS minus a small (second derived) neighbourhood of Y. Therefore $\mathrm{Cl}(nB)=nB+X$ consists of the disjoint union of a finite number of little stars of edges (one for each vertex of $\mathrm{Cl}(nS)$), and $\mathrm{Cl}(fB)$ is isomorphic to $\mathrm{Cl}(nB)$ by projection from V.

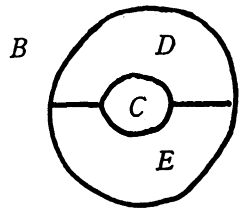
The second stage of the proof consists of constructing a 3-ball C in the interior of B, such that fB, nB lie in the interior, exterior of C, respectively. First choose a tree T, piecewise linearly embedded in the interior of B, that contains $\mathrm{Cl}(fB)$ and does not meet nB; to construct T, observe that $\mathrm{Cl}(fB)\subset\mathrm{Cl}(fS)\subset$the interior of B, and so join up each of the components of $\mathrm{Cl}(fB)$ by arcs to a base point in the interior of B. Let B_1 be a subdivision of B containing subcomplexes triangulating $\mathrm{Cl}(nB)$, $\mathrm{Cl}(fB)$, T, and let C_0 be the closed simplicial neighbourhood of T in the second derived complex B_1'' of B_1. Then C_0 is a 3-ball because T is collapsible. T is contained in the interior of C_0, and C_0 is contained in the interior of B. Unfortunately C_0 is not the ball that we require because it meets nB (for it contains the closed star in B of each point $x \in X \subset \mathrm{Cl}(nB)$). In order to obtain the 3-ball C with the desired properties, we have to snip off at each $x \in X$ the offending knob of C_0 which butts into nB, as follows:



Choose $x \in X$, and let $J = \mathrm{link}(x, B_1'')$, which is a 2-sphere. Let $J_1 = J \cap nB$, which is a finite set of points, and let $J_2 = J \cap \mathrm{Cl}(C_0 - xJ)$, which is a finite disjoint set of disks (being a second derived neighbourhood of the finite set of points $J \cap T$). Let K be a disk in J, such that J_1, J_2 lie in the interior, exterior of K, respectively. Then xK is a 3-ball contained in C_0, whose boundary $K + x\dot{K}$ meets $\dot{C}_0$ in the disk K. Therefore $\mathrm{Cl}(C_0 - xK)$ is a 3-ball, by [**1**, Corollary 14: 5b]. We have snipped off the knob of C_0 at x which butts into nB at x. At the same time, since fB does not meet xK, we have kept fB in the interior of $\mathrm{Cl}(C_0 - xK)$. Proceeding inductively for all the vertices of X, we are eventually left with a 3-ball C, not meeting nB, and whose interior contains fB.

Since $C \subset C_0 \subset$ interior of B, the closure of the complement $\mathrm{Cl}(B - C)$ is an annulus. In other words, if Δ is a 3-simplex and I the unit interval, then there is a piecewise linear homeomorphism from $\dot{\Delta} \times I$ onto $\mathrm{Cl}(B - C)$, by [**3**, Theorem 3]. Let a be a vertex of Δ, and Γ the opposite face; let D, E be the 3-balls in $\mathrm{Cl}(B - C)$ that are the images under the homeomorphism of $\Gamma \times I$, $a\dot{\Gamma} \times I$, respectively. Then $C + D$ is also a 3-ball, and $B = C + D + E$.

The second stage of the proof is complete. We are now ready to complete the proof of the theorem by making two more cellular moves. The cones VD, VE are 4-balls, whose interiors do not meet



B, because $D+E=\mathrm{Cl}(B-C)$ consists only of near- or non-singular points of B. The cone VC is a 4-ball, because C consists only of far- or non-singular points of B. Therefore we have the cellular moves

$$\begin{aligned} S &\sim V\dot{B} + B, \text{ across } VA, \\ &\sim V(C+D)^{\cdot} + (C+D), \text{ across } VE, \\ &\sim V\dot{C} + C, \text{ across } VD, \\ &= (VC)^{\cdot}, \text{ the boundary of a 4-ball.} \end{aligned}$$

Hence S is unknotted, and the theorem is proved.

Isotopy. The result can be restated in terms of isotopy in various ways (depending on which definition of isotopy is chosen). We give one such statement as a Corollary; it can be deduced fairly easily from the equivalent definition of unknotting by simplicial moves (see [6, Theorem 1]), and the proof is given in [8].

Choose a fixed 4-simplex in E^6, and let Σ denote its boundary. Let $\Sigma^{(r)}$ denote the rth barycentric derived complex of Σ. We call a map $\Sigma \to E^6$ *linear on* $\Sigma^{(r)}$ if it maps each simplex of $\Sigma^{(r)}$ linearly into E^6. By an *isotopy* of Σ, we mean a piecewise linear map $F: I \times \Sigma \to E^6$, such that for each t, $0 \leqq t \leqq 1$, the map $F_t: \Sigma \to E^6$ given by $F_t(x) = F(t, x)$ is an embedding.

COROLLARY. *Given a combinatorial 3-sphere S piecewise linearly embedded in E^6, then there exists an integer r and an isotopy $F: I \times \Sigma \to E^6$, such that $F_0(\Sigma) = S$, F_1 is the identity, and F_t is linear on $\Sigma^{(r)}$ for all t, $0 \leqq t \leqq 1$.*

REFERENCES

1. J. W. Alexander, *The combinatorial theory of complexes*, Ann. of Math (2) **31** (1930), 292–320.

2. A. Haefliger, *Knotted* $(4k-1)$*-spheres in* $6k$*-space*, Ann. of Math. (2) **75** (1962), 452–466.

3. M. H. A. Newman, *On the superposition of n-dimensional manifolds*, J. London Math. Soc. **2** (1927), 56–64.

4. J. H. C. Whitehead, *Simplicial spaces, nuclei, and m-groups*, Proc. London Math. Soc. **45** (1939), 243–327.

5. E. C. Zeeman, *Unknotting spheres in five dimensions*, Bull. Amer. Math. Soc. **66** (1960), 198.
6. ———, *Unknotting spheres*, Ann. of Math. (2) **72** (1960), 350–361.
7. ———, *Unknotting combinatorial balls*, (to appear).
8. ———, *Isotopies of manifolds*, (to appear).

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A WEAK TYCHONOFF THEOREM AND THE AXIOM OF CHOICE

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1. A well-known result of Kelley [1] asserts that the Tychonoff theorem (the product of compact spaces is compact) implies the axiom of choice, establishing the equivalence of these propositions. In what follows we show that an apparently weaker form of the Tychonoff theorem also implies the axiom of choice. The proof is quite brief and direct.

WEAK TYCHONOFF THEOREM. *The product of a family of mutually homeomorphic compact spaces is compact.*

PROPOSITION. *The weak Tychonoff theorem implies the axiom of choice.*

PROOF. Let $\mathfrak{A}$ be a disjoint family of nonempty sets and let Σ be the union of the members of $\mathfrak{A}$. If $\Sigma^{\mathfrak{A}}$ denotes the set of functions on $\mathfrak{A}$ into Σ, we shall demonstrate the existence of an element of $\Sigma^{\mathfrak{A}}$ which is a choice function, i.e., a function which maps each A in $\mathfrak{A}$ into an element of itself.

Topologize Σ by defining a subset U of Σ to be open if U is the empty set or if $\Sigma - U$ is the union of finitely many members of $\mathfrak{A}$. Since $\mathfrak{A}$ is a disjoint family this collection of open sets clearly satisfies the axioms for a topology, and it is easily seen to be a compact topology. By the weak Tychonoff theorem, $\Sigma^{\mathfrak{A}}$ is also compact.

For each $A \in \mathfrak{A}$ let F_A denote the set of all $f \in \Sigma^{\mathfrak{A}}$ such that $f(A)$

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