Theorem 1 is measured by the identification space M^* whose elements are the leaves of N and the components of $M-N$. If M^* is metrizable, it can be shown to have inductive dimension 1. In this case the argument above can be replaced by an application of the Vietoris mapping theorem.

Bibliography

1. W. Ambrose and I. M. Singer, *A theorem on holonomy*, Trans. Amer. Math. Soc. **75** (1953), 428–443.

2. E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, Gauthier-Villars, Paris, 1951.

3. S. Chern and R. Lashof, *On the total curvature of immersed manifolds*, Amer. J. Math. **79** (1957), 306–318.

4. P. Hartman and L. Nirenberg, *On spherical image maps whose Jacobians do not change sign*, Amer. J. Math. **81** (1959), 901–920.

University of California, Los Angeles

ON THE EMBEDDABILITY OF THE REAL PROJECTIVE SPACES[1]

MARK MAHOWALD[2]

In a paper of the same title, Massey [4] proved that if $2^{k-1}+2^{k-2}-1\leqq n<2^k$ then P_n cannot be differentiably embedded in R^{2^k}. By using the technique of Massey in a different way we can prove the following theorem which clearly includes Massey's.

Theorem. *If $2^{k-1}<n<2^k$ then P_n cannot be embedded differentiably in Euclidean space of dimension 2^k.*

Besides the result of Massey, the main result in this direction is if $2^{k-1}<n<2^k$ then P_n cannot be embedded differentiably in R^{2^k-1}. Our result yields, in particular, that for P_{2^k+1}, the embedding in $R^{2^{k+1}+1}$ given by Hopf and James [1] is the best possible.

The following information from [3; 4] will be needed. Let M be a n-manifold differentiably embedded in R^{n+k+1}; and let $p: E\to M$ denote the bundle of unit normal vectors. Then there exist subalgebras $A^*(E, Z)\subset H^*(E, Z)$ and $A^*(E, Z_2)\subset H^*(E, Z_2)$ which satisfy the following conditions:

1. $A^0(E, G)=H^0(E, G)$,
2. $H^q(E, G)=A^q(E, G)+p^*(H^q(B, G))$ $(0<q<n+k)$,
3. $A^q(E, G)=0$, $q\geqq n+k$,

Received by the editors October 2, 1961.

[1] Sponsored by the U. S. Army Research Office (Durham).

[2] The referee has informed me that the result of this paper has been obtained independently by Mr. J. P. Levine in his thesis at Princeton University.

where $G=Z$ or Z_2. Moreover the algebra A^* is closed under all natural cohomology operations.

By the well-known theorem of Seifert and Whitney, the characteristic class of a normal bundle vanishes hence the Gysin sequence breaks up into parts of length three, $0\to H^{i+k}(M)\to^{p^*} H^{i+k}(E) \to^{\psi} H^i(M)\to 0$. Because of (2) ψ must be an isomorphism on A^*. Each element in $H^*(E)$ can be written in the form $p^*(b_1)+a\cdot p^*(b_2)$ where a is the unique element in A^k such that $\psi(a)=1$. For simplicity we will suppress the map p^* and write a general element in $H^*(E)$ as $b_1+a\cdot b_2$.

The proof of the theorem will consist in showing that P_m $(m=2^{k-1}+1)$ cannot be differentiably embedded in R^{2^k}. The result will then follow from the fact that P_m can be differentiably embedded in P_n for $n>2^{k-1}$.

A simple computation shows that the Stiefel-Whitney classes for the tangent bundle for $P_{2^{k-1}+1}$ are as follows: $W_0=1$, $W_2=\alpha^2$, $W_{m-1}=\alpha^{m-1}$ where α is the nonzero element of $H^1(P_m, Z_2)$ and all other $W_i=0$. Using the fact that $W\cdot\overline{W}=1$ we see that $\overline{W}_{2i}=\alpha^{2i}$, $0\leqq i\leqq(m-3)/2$ and $\overline{W}_j=0$ for all other j.

Suppose P_m is differentiably embedded in R^{2^k}. Let E be the bundle of unit normal vectors over P_m for this embedding. Then E is a $(m-3)$-sphere bundle and since $\overline{W}_{m-2}=0$, the characteristic class vanishes. Let $a\in A^{m-3}$ be the element such that $\psi(a)=1$. Suppose that the nonzero element of A^{m-2} is of the form $\alpha^{m-2}+a\cdot\alpha$; then, since $Sq^1(\alpha^{m-2}+a\cdot\alpha)=\alpha^{m-1}+a\cdot\alpha^2$ because a is an integer class, we have that the nonzero element of A^{m-1} must be $\alpha^{m-1}+a\cdot\alpha^2$. But $(\alpha^{m-2}+a\cdot\alpha)(\alpha^{m-1}+a\cdot\alpha^2)=a\cdot\alpha^m+a^2\cdot\alpha^3+a\cdot\alpha^m$. This element must be in A^{2m+3} which is zero by (3).

But it equals $a^2\cdot\alpha^3=(Sq^{m-3}a)\cdot\alpha^3=a\cdot\overline{W}_{m-3}\cdot\alpha^3$ by a result due to Liao [2], and this equals $a\cdot\alpha^m\neq 0$. Hence the nonzero element of A^{m-2} is of the form $a\cdot\alpha$. But then $Sq^1a\cdot\alpha=a\cdot\alpha^2$ is the nonzero element of A^{m-1} and $(a\cdot\alpha^2)(a\cdot\alpha^1)\in A^{2m-3}=0$ but it, as before, equals $a\cdot\alpha^m$ which is not zero.

References

1. I. M. James, *Some embeddings of projective spaces*, Proc. Cambridge Philos. Soc. **55** (1959), 294–298.

2. S. D. Liao, *On the theory of obstructions of fiber bundles*, Ann. of Math. (2) **60** (1954), 146–191.

3. W. S. Massey, *On the cohomology ring of a sphere bundle*, J. Math. Mech. **7** (1958), 265–290.

4. ———, *On the embeddability of the real projective spaces*, Pacific J. Math. **9** (1959), 783–789.

Syracuse University