ON f-RINGS WITH THE ASCENDING CHAIN CONDITION1

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Introduction. In [1] Birkhoff and Pierce obtain the structure of f-rings2 which have no nonzero nilpotent elements and satisfy the descending chain condition for l-ideals. More recently, D. G. Johnson [4] gives the structure of J-semi-simple f-rings (§2) with the descending chain condition for l-ideals. In this note our principal aim is to give the structure of f-rings with various ascending chain conditions. We first show (Theorem 1) that in f-rings the ascending and descending chain conditions for closed l-ideals are equivalent and that an f-ring with these conditions can be characterized as a subdirect sum of finitely many totally ordered rings. Next (Theorem 2) we specialize to the case of f-rings with no nonzero nilpotent elements. In §2 we consider J-semi-simple f-rings. For these f-rings we show (Theorem 4) that the ascending and descending chain conditions for l-ideals and for closed l-ideals are all equivalent.

In [3] Goldie proves that a semi-simple ring with the ascending chain condition for ideals is a subdirect sum of a finite number of semi-simple prime rings. An examination of the proof of this result shows that he proves even more, namely, that a semi-prime ring with the ascending chain condition for annihilator ideals is a subdirect sum of a finite number of prime rings. The results of this note provide f-ring analogues of the results of [3], and the techniques we employ are patterned after those of Goldie.

1. Chain conditions for closed l-ideals. Let A be an f-ring. By an l-ideal of A we mean a ring ideal I such that for all a, bEA if bEI and |a| ≤ |b|, then a∈I. If S is a nonempty subset of A, then we set

\[ S^l = \{ a \in A ; \ a \wedge x = 0 \ (x \in S) \} . \]

It is clear that: (i) \( S^l \) is an l-ideal of A; (ii) \( S \cap S^l = \{ 0 \} \); (iii) \( S \subseteq S^{l-1} \); and (iv) \( S^\perp \) is contained in both the left and right (ring) annihilators of S. We say that S is complemented in case \( S^l \not= \{ 0 \} \) and closed in case \( S = S^{l-1} \).

1 This work was supported by a grant from the National Science Foundation.
2 An f-ring is a lattice-ordered ring in which \( a \wedge b = 0 \) and \( c \geq 0 \) imply \( a \wedge b = ac \wedge b = 0 \). In [1] Birkhoff and Pierce, who introduced the concept, prove that f-rings are characterizable as subdirect sums of totally ordered rings. For the general theory of lattice-ordered rings and of f-rings see Birkhoff and Pierce [1], Johnson [4], and Pierce [5].

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Lemma 1. If $I$ is a nonzero $l$-ideal of an $f$-ring $A$, then the following statements are equivalent:

1. $I$ is totally ordered (as a sub $f$-ring of $A$);
2. $I^\perp$ is a maximal closed $l$-ideal;
3. $A/I^\perp$ is totally ordered.

Proof. (1) $\Rightarrow$ (3). Assume that $I$ is totally ordered. Then if $a, b \in I^\perp$ are positive, there exists an element $c \in I$ such that $a \wedge c > 0$ and $b \wedge c > 0$. Since $I$ is totally ordered and an $l$-ideal of $A$, $(a \wedge b) \wedge c > 0$. Thus $a \wedge b \in I^\perp$, and so, $A/I^\perp$ is totally ordered.

(3) $\Rightarrow$ (2). Assume $A/I^\perp$ is totally ordered. To see that $I^\perp$ is a maximal closed $l$-ideal, it will suffice to show that if $a \in I^\perp$, then $\{a\} \cup I^\perp = \{0\}$ since the closed $l$-ideal generated by $\{a\} \cup I^\perp$ is

$$(\{a\} \cup I^\perp)^\perp = (\{a\} \cap I^\perp)^\perp.$$ 

But if $x \in \{a\} \cap I^\perp$, then $A/I^\perp$ is totally ordered, since $|x| \wedge |a| = 0$, and since $a \in I^\perp$, we have $x \in I^\perp$. Therefore $\{a\} \cap I^\perp \subseteq I^\perp \cap I^\perp = \{0\}$.

(2) $\Rightarrow$ (1). If $I$ is not totally ordered, then $I^\perp$ is not totally ordered. So there exist nonzero elements $a, b \in I^\perp$ such that $a \wedge b = 0$. If $J$ is the $l$-ideal generated by $I^\perp \cup \{a\}$, then $b \in J^\perp$. Thus $J^\perp$ is a proper closed $l$-ideal properly containing $I^\perp$; hence, $I^\perp$ is not a maximal closed $l$-ideal.

Lemma 2. If $M$ and $N$ are maximal closed $l$-ideals of an $f$-ring $A$, then $M \neq N$ if and only if $N^\perp \neq M$.

Proof. By Lemma 1, $N^\perp$ is totally ordered. Since $M$ is closed, it is clear then that either $N^\perp \subseteq M$ or $N^\perp \supseteq M$. So if $N^\perp \not\subseteq M$, then $M \subseteq N$ and, by the maximality of $M$, $M = N$. Conversely, if $M = N$, then $N^\perp = M^\perp \subseteq M$ since $M^\perp \neq \{0\}$.

In general, an $f$-ring need not have any maximal closed $l$-ideals. An example of such an $f$-ring is the $f$-ring of all continuous real-valued functions on $[0, 1]$. Also, a maximal closed $l$-ideal need not be a maximal $l$-ideal. For example, let $Q[x]$ be the ring of polynomials in one indeterminate over the rational field ordered lexicographically $(1 > \lambda > \lambda^2 > \cdots)$. Then $\{0\}$ is a maximal closed $l$-ideal but not a maximal $l$-ideal.

Lemma 3. If $A$ is an $f$-ring satisfying the ascending chain condition for closed $l$-ideals, then every complemented $l$-ideal of $A$ is contained in a maximal closed $l$-ideal.

$^2$ See Johnson [4, p. 172].
PROOF. If \( I \) is a complemented \(-\)ideal, then \( I^\perp \) is a proper closed \(-\)ideal containing \( I \).

**Lemma 4.** If \( A \) is an \( f \)-ring satisfying the ascending chain condition for closed \(-\)ideals, then the set \( \mathfrak{M} \) of maximal closed \(-\)ideals of \( A \) is finite and \( \bigcap \mathfrak{M} = \{0\} \).

**Proof.** By Lemma 3, \( \mathfrak{M} \neq \emptyset \). We show first that \( \bigcap \mathfrak{M} = \{0\} \). For if \( \bigcap \mathfrak{M} \neq \{0\} \), then \( (\bigcap \mathfrak{M})^\perp \) is complemented. Thus, by Lemma 3, there is an \( M \in \mathfrak{M} \) with \( (\bigcap \mathfrak{M})^\perp \subseteq M \). Since this implies \( M^\perp \subseteq M \), we have the contradiction \( M^\perp = \{0\} \); hence \( \bigcap \mathfrak{M} = \{0\} \).

Now using the ascending chain condition for closed \(-\)ideals, we see that there exist \( M_1, \ldots, M_n \in \mathfrak{M} \) such that

\[
M^\perp \subseteq (M_1 \cap \cdots \cap M_n)^\perp \quad (M \in \mathfrak{M}).
\]

Thus, \( M_1 \cap \cdots \cap M_n = \{0\} \). If \( M \in \mathfrak{M} \) and \( M \not\in M_i \) (\( i = 1, \ldots, n \)), then, by Lemma 2, \( M^\perp \subseteq M_1 \cap \cdots \cap M_n \) contrary to \( M^\perp \neq \{0\} \). Therefore, \( \mathfrak{M} = \{M_1, \ldots, M_n\} \).

**Theorem 1.** For an \( f \)-ring \( A \) the following statements are equivalent:

1. \( A \) has the ascending chain condition for closed \(-\)ideals.
2. \( A \) has the descending chain condition for closed \(-\)ideals.
3. \( A \) is isomorphic to a subdirect sum of a finite number of totally ordered rings.

**Proof.** The equivalence of (1) and (2) is clear since the mapping \( I \rightarrow I^\perp \) is a dual automorphism of the lattice of closed \(-\)ideals of \( A \).

The implication \( (3) \Rightarrow (1) \) is trivial. Finally, the implication \( (1) \Rightarrow (3) \) follows immediately from Lemmas 1 and 4.

If \( A \) is an \( f \)-ring, then the set \( N(A) \) of all nilpotent elements of \( A \) is an \(-\)ideal called the \(-\)radical of \( A \) [1]. Clearly, \( N(A/N(A)) = \{0\} \) and \( N(I) = N(A) \cap I \) for any \(-\)ideal \( I \) of \( A \). The \(-\)radical \( N(A) \) of \( A \) can also be characterized [5] as the intersection of all prime \(-\)ideals of \( A \). Recall [4] that in an \( f \)-ring \( A \) an \(-\)ideal \( P \) is prime if and only if for all \( a, b \in A \), \( ab \in P \) implies \( a \in P \) or \( b \in P \). This is also equivalent to the property: \( A/P \) is totally ordered with no nonzero divisors of zero.

If \( A \) is an \( f \)-ring with \( N(A) = \{0\} \), then it follows that if \( S \subseteq A \) is nonvoid, its left annihilator, right annihilator, and \( S^\perp \) coincide [1, p. 63].

**Lemma 5.** Let \( A \) be an \( f \)-ring with \( N(A) = \{0\} \) and let \( P \) be an \(-\)ideal of \( A \). Then \( P \) is a complemented prime \(-\)ideal if and only if \( P \) is a maximal closed \(-\)ideal.
Proof. If $P$ is a complemented prime $l$-ideal, then $P^\bot \neq 0$, whence $P^\bot \subseteq P$. Since $P^\bot P^\bot = \{0\}$, we have that $P^\bot \subseteq P$, so that $P$ is closed. Also, since $P$ is prime, $A/P$ is totally ordered. Therefore, by Lemma 1 (with $I = P^\bot$), we have that $P = P^\bot$ is a maximal closed $l$-ideal.

Conversely, it will suffice to show that if $P$ is maximal closed, then it is prime. But in this case $P^\bot$ is totally ordered by Lemma 1, so that if $a, b \in P$, there exists a $c \in P^\bot$ such that $|a| \cap |c| \neq 0$ and $|b| \cap |c| \neq 0$. Now $N(P^\bot) = N(A) \cap P^\bot = \{0\}$, so that $P^\bot$ is a prime $f$-ring. Therefore, since $|a| \cap |c|, |b| \cap |c| \in P^\bot$, we have

$$0 \neq (|a| \cap |c|)(|b| \cap |c|) \subseteq |a| \cap |b| = |ab|.$$ 

Since $P^\bot \cap P = \{0\}$, it follows that $(|a| \cap |c|)(|b| \cap |c|) \in P$. Thus $ab \in P$, and $P$ is prime.

Now from Lemma 4, Lemma 5, and Theorem 1 we readily conclude

Theorem 2. Let $A$ be an $f$-ring with $N(A) = \{0\}$. Then the following statements are equivalent:

1. $A$ has the ascending chain condition for closed $l$-ideals.
2. $A$ has the descending chain condition for closed $l$-ideals.
3. $A$ is isomorphic to a subdirect sum of a finite number of totally ordered rings having no nonzero divisors of zero.

As we have now seen the ascending and descending chain conditions for closed $l$-ideals are equivalent in any $f$-ring. However, even for $f$-rings with zero $l$-radical the ascending and descending chain conditions for $l$-ideals need not be equivalent. For example, the $f$-ring $Q[\lambda]$, which has zero $l$-radical, satisfies the ascending but not the descending chain condition for $l$-ideals. Note, however, that if $N(A) = \{0\}$ and if $A$ satisfies the descending chain condition for $l$-ideals, then $A$ is isomorphic to a finite direct sum of $l$-simple totally ordered rings [1, Theorem 17] and therefore satisfies the ascending chain condition for $l$-ideals.

We also observe that in Theorems 1 and 2 the "subdirect sum" of statement (3) cannot be strengthened to "direct sum." For let $A$ be the sub-$f$-ring of the direct sum of two copies of $Q[\lambda]$ defined by

$$A = \{(f, g); f, g \in Q[\lambda] \text{ with } f(0) = g(0)\}.$$ 

Then $N(A) = \{0\}$ and $A$ has the ascending chain condition for closed $l$-ideals, but $A$ cannot be isomorphic to a direct sum of totally ordered rings.

2. Chain conditions in $J$-semi-simple $f$-rings. An $l$-ideal $P$ of an $f$-ring $A$ is $l$-primitive if and only if $A/P$ is an $l$-simple ordered ring.
with identity. Thus an \( l \)-primitive \( l \)-ideal is prime. The \( J \)-radical, \( J(A) \), of \( A \) is the intersection of all \( l \)-primitive \( l \)-ideals of \( A \). Clearly \( N(A) \subseteq J(A) \). If \( J(A) = \{0\} \), then \( A \) is \( J \)-semi-simple.

The example \( \mathbb{Q}[x] \) shows that in \( f \)-rings with zero \( l \)-radical closed prime \( l \)-ideals need not be \( l \)-primitive. However, for \( J \)-semi-simple \( f \)-rings we have

**Lemma 6.** If \( P \) is a closed prime \( l \)-ideal of a \( J \)-semi-simple \( f \)-ring \( A \), then \( P \) is an \( l \)-primitive \( l \)-ideal and

\[
A = P \oplus P^\perp.
\]

**Proof.** By Lemmas 1 and 5, \( P^\perp \) is totally ordered and so, since \( J(P^\perp) = J(A) \cap P^\perp = \{0\} \) \([4, p. 188]\), \( P^\perp \) is an \( l \)-primitive \( f \)-ring. Let \( e \in P^\perp \) be the identity for \( P^\perp \). Then, since \( P \) is the right ring annihilator of \( P^\perp \),

\[
a = (a - ea) + ea \in P + P^\perp
\]

for all \( a \in A \). Thus, \( A = P + P^\perp \) and since \( P \cap P^\perp = \{0\} \), this sum is direct. Therefore \( A/P \) is isomorphic to \( P^\perp \) and \( P \) is an \( l \)-primitive \( l \)-ideal.

**Theorem 3.** Let \( A \) be a \( J \)-semi-simple \( f \)-ring satisfying the ascending chain condition for closed \( l \)-ideals. Then the set \( \Phi \) of closed prime \( l \)-ideals of \( A \) coincides with the set of \( l \)-primitive \( l \)-ideals of \( A \), and \( A \) is the direct sum of the \( l \)-ideals \( P_i \) (\( P \subseteq \Phi \)).

**Proof.** By Lemmas 4 and 5, \( \Phi \) is finite. Let \( \Phi = \{P_1, \ldots , P_n\} \). By Lemmas 2 and 6 the sum

\[
P_1^\perp + \cdots + P_n^\perp
\]

is direct and each \( P_i^\perp \) is an \( l \)-primitive \( f \)-ring. If \( e_i \in P_i^\perp \) is the identity of \( P_i^\perp \), then for each \( a \in A \)

\[
a - \sum_{i=1}^{n} e_i a \in P_1 \cap \cdots \cap P_n.
\]

By Lemma 4 this implies that

\[
a = \sum_{i=1}^{n} e_i a,
\]

whence

\[
A = P_1^\perp \oplus \cdots \oplus P_n^\perp.
\]

\(^4\) The notions of \( l \)-primitivity and of the \( J \)-radical as well as the structure theory of \( J \)-semi-simple \( f \)-rings are due to Johnson [4].
To complete the proof it will suffice, in view of Lemma 6, to show that each \( l \)-primitive \( l \)-ideal \( P \) of \( A \) is one of the \( P_i \) (\( i = 1, \ldots, n \)). But since \( P \) is proper \( P_i \subsetneq P \) for some \( i \) and so, since \( P \) is prime \( P_i \subsetneq P \). However, \( P_i \) is \( l \)-primitive, hence maximal [4, p. 187]; thus \( P_i = P \).

**Theorem 4.** For a \( J \)-semisimple \( f \)-ring \( A \) the following statements are equivalent:

1. \( A \) has the ascending chain condition for closed \( l \)-ideals.
2. \( A \) has the descending chain condition for closed \( l \)-ideals.
3. \( A \) has the ascending chain condition for \( l \)-ideals.
4. \( A \) has the descending chain condition for \( l \)-ideals.
5. \( A \) is isomorphic to the direct sum of a finite set of \( l \)-simple totally ordered rings with identity.

**Proof.** The implication \((1) \Rightarrow (5)\) is by Theorem 3. Also \((1) \Leftrightarrow (2)\) by Theorem 1. By Theorem II.5.8 of [4] we have \((4) \Leftrightarrow (5)\). Since \((5) \Rightarrow (3) \Rightarrow (1)\) are trivial, the proof is complete.

**3. Remarks.** If an \( f \)-ring \( A \) satisfies the ascending chain condition for \( l \)-ideals (closed \( l \)-ideals), then each \( l \)-ideal (closed \( l \)-ideal) of \( A \) is principal. For the \( l \)-ideal (closed \( l \)-ideal) generated by \( \{a_1, \ldots, a_n\} \) is also generated by \( |a_1| \lor \ldots \lor |a_n| \). Conversely, if each \( l \)-ideal of an \( f \)-ring \( A \) is principal, then \( A \) satisfies the ascending chain condition for \( l \)-ideals. Such a converse is not valid, however, for closed \( l \)-ideals. For example, in the \( f \)-ring of all real-valued functions on the integers every closed \( l \)-ideal is principal, but this \( f \)-ring clearly does not have the ascending chain condition for closed \( l \)-ideals.

If \( A \) is an arbitrary ring with the descending chain condition for right ideals, then \( A \) has the ascending chain condition for right ideals if and only if the additive group of \( A \) contains no \( p^\infty \) group (Fuchs [2]). Certainly the additive group of an \( f \)-ring has this property since this group must be torsion free. However, let \( A \) be the ring whose additive group is that of \( \mathbb{Q}[\lambda] \) and with multiplication defined by

\[
\left( \sum_{i=0}^{m} a_i \lambda^i \right) \left( \sum_{j=0}^{n} b_j \lambda^j \right) = \sum_{i=0}^{m} \left( \sum_{j=0}^{n} (a_i b_j) \lambda^i \right).
\]

Order \( A \) by

\[
a_0 + a_1 \lambda + \cdots + a_m \lambda^m > 0
\]

in case \( a_0 > 0 \) or \( a_0 = 0 \) and \( a_m > 0 \). Then \( A \) is a commutative \( f \)-ring with identity which satisfies the descending but not the ascending chain condition for \( l \)-ideals.

Finally, it is known [4, p. 213] that in an \( f \)-ring with zero \( l \)-radical
the descending chain condition on $l$-ideals and the descending chain condition on right $l$-ideals are equivalent. The corresponding statement for ascending chain conditions fails. For it can be shown that in an example due to Johnson [4, pp. 208–209] we have an $f$-ring with zero $l$-radical which satisfies the ascending chain condition for $l$-ideals but not the ascending chain condition for right $l$-ideals.\(^6\)

**References**


\(^6\) In the $f$-ring of this example the principal right $l$-ideals generated by the elements $xa, x^2a, \cdots$ form a properly ascending chain.