ON f-RINGS WITH THE ASCENDING CHAIN CONDITION[1]

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Introduction. In [1] Birkhoff and Pierce obtain the structure of f-rings[2] which have no nonzero nilpotent elements and satisfy the descending chain condition for l-ideals. More recently, D. G. Johnson [4] gives the structure of J-semi-simple f-rings (§2) with the descending chain condition for l-ideals. In this note our principal aim is to give the structure of f-rings with various ascending chain conditions. We first show (Theorem 1) that in f-rings the ascending and descending chain conditions for closed l-ideals are equivalent and that an f-ring with these conditions can be characterized as a subdirect sum of finitely many totally ordered rings. Next (Theorem 2) we specialize to the case of f-rings with no nonzero nilpotent elements. In §2 we consider J-semi-simple f-rings. For these f-rings we show (Theorem 4) that the ascending and descending chain conditions for l-ideals and for closed l-ideals are all equivalent.

In [3] Goldie proves that a semi-simple ring with the ascending chain condition for ideals is a subdirect sum of a finite number of semi-simple prime rings. An examination of the proof of this result shows that he proves even more, namely, that a semi-prime ring with the ascending chain condition for annihilator ideals is a subdirect sum of a finite number of prime rings. The results of this note provide f-ring analogues of the results of [3], and the techniques we employ are patterned after those of Goldie.

1. **Chain conditions for closed l-ideals.** Let A be an f-ring. By an *l-ideal* of A we mean a ring ideal I such that for all a, $b\in A$ if $b\in I$ and $|a|\leqq|b|$, then $a\in I$. If S is a nonempty subset of A, then we set

$$S^{\perp} = \{a \in A;\ |a| \wedge |x| = 0\ (x \in S)\}.$$

It is clear that: (i) $S^{\perp}$ is an l-ideal of A; (ii) $S\cap S^{\perp}=\{0\}$; (iii) $S\subseteq S^{\perp\perp}$; and (iv) $S^{\perp}$ is contained in both the left and right (ring) annihilators of S. We say that S is *complemented* in case $S^{\perp}\neq\{0\}$ and *closed* in case $S=S^{\perp\perp}$.

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[2] An *f-ring* is a lattice-ordered ring in which $a\wedge b=0$ and $c\geqq 0$ imply $ca\wedge b=ac\wedge b=0$. In [1] Birkhoff and Pierce, who introduced the concept, prove that *f-rings* are characterizable as subdirect sums of totally ordered rings. For the general theory of lattice-ordered rings and of f-rings see Birkhoff and Pierce [1], Johnson [4], and Pierce [5].

LEMMA 1. *If I is a nonzero l-ideal of an f-ring A, then the following statements are equivalent*:

(1) *I is totally ordered* (*as a sub f-ring of A*);

(2) *$I^{\perp}$ is a maximal closed l-ideal*;

(3) *$A/I^{\perp}$ is totally ordered.*

PROOF. (1)$\Rightarrow$(3). Assume that I is totally ordered. Then if $a, b \notin I^{\perp}$ are positive, there exists an element $c \in I$ such that $a \wedge c > 0$ and $b \wedge c > 0$. Since I is totally ordered and an l-ideal of A, $(a \wedge b) \wedge c > 0$. Thus $a \wedge b \notin I^{\perp}$, and so, $A/I^{\perp}$ is totally ordered.

(3)$\Rightarrow$(2). Assume $A/I^{\perp}$ is totally ordered. To see that $I^{\perp}$ is a maximal closed l-ideal, it will suffice to show that if $a \notin I^{\perp}$, then $\{a\}^{\perp} \cap I^{\perp\perp} = \{0\}$ since the closed l-ideal generated by $\{a\} \cup I^{\perp}$ is

$$(\{a\} \cup I^{\perp})^{\perp\perp} = (\{a\}^{\perp} \cap I^{\perp\perp})^{\perp}.$$

But if $x \in \{a\}^{\perp} \cap I^{\perp\perp}$, then since $A/I^{\perp}$ is totally ordered, since $|x| \wedge |a| = 0$, and since $a \notin I^{\perp}$, we have $x \in I^{\perp}$. Therefore $\{a\}^{\perp} \cap I^{\perp\perp} \subseteq I^{\perp} \cap I^{\perp\perp} = \{0\}$.

(2)$\Rightarrow$(1). If I is not totally ordered, then $I^{\perp\perp}$ is not totally ordered. So there exist nonzero elements $a, b \in I^{\perp\perp}$ such that $a \wedge b = 0$. If J is the l-ideal generated by $I^{\perp} \cup \{a\}$, then $b \in J^{\perp}$. Thus $J^{\perp\perp}$ is a proper closed l-ideal properly containing $I^{\perp}$; hence, $I^{\perp}$ is not a maximal closed l-ideal.

LEMMA 2. *If M and N are maximal closed l-ideals of an f-ring A, then $M \neq N$ if and only if $N^{\perp} \subseteq M$.*

PROOF. By Lemma 1, $N^{\perp}$ is totally ordered. Since M is closed, it is clear then that either $N^{\perp} \subseteq M$ or $N^{\perp} \subseteq M^{\perp}$. So if $N^{\perp} \not\subseteq M$, then $M \subseteq N$ and, by the maximality of M, $M = N$. Conversely, if $M = N$, then $N^{\perp} = M^{\perp} \not\subseteq M$ since $M^{\perp} \neq \{0\}$.

In general, an f-ring need not have any maximal closed l-ideals. An example of such an f-ring is the f-ring of all continuous real-valued functions on $[0, 1]$. Also, a maximal closed l-ideal need not be a maximal l-ideal. For example, let $Q[\lambda]$ be the ring of polynomials in one indeterminate over the rational field ordered lexicographically ($1 > \lambda > \lambda^2 > \cdots$).[3] Then $\{0\}$ is a maximal closed l-ideal but not a maximal l-ideal.

LEMMA 3. *If A is an f-ring satisfying the ascending chain condition for closed l-ideals, then every complemented l-ideal of A is contained in a maximal closed l-ideal.*

[3] See Johnson [4, p. 172].

PROOF. If I is a complemented l-ideal, then $I^{\perp\perp}$ is a proper closed l-ideal containing I.

LEMMA 4. *If A is an f-ring satisfying the ascending chain condition for closed l-ideals, then the set $\mathfrak{M}$ of maximal closed l-ideals of A is finite and $\bigcap\mathfrak{M} = \{0\}$.*

PROOF. By Lemma 3, $\mathfrak{M} \neq \varnothing$. We show first that $\bigcap\mathfrak{M} = \{0\}$. For if $\bigcap\mathfrak{M} \neq \{0\}$, then $(\bigcap\mathfrak{M})^\perp$ is complemented. Thus, by Lemma 3, there is an $M \in \mathfrak{M}$ with $(\bigcap\mathfrak{M})^\perp \subseteq M$. Since this implies $M^\perp \subseteq M$, we have the contradiction $M^\perp = \{0\}$; hence $\bigcap\mathfrak{M} = \{0\}$.

Now using the ascending chain condition for closed l-ideals, we see that there exist $M_1, \cdots, M_n \in \mathfrak{M}$ such that

$$M^\perp \subseteq (M_1 \cap \cdots \cap M_n)^\perp \qquad (M \in \mathfrak{M}).$$

Thus, $M_1 \cap \cdots \cap M_n = \{0\}$. If $M \in \mathfrak{M}$ and $M \neq M_i$ $(i = 1, \cdots, n)$, then, by Lemma 2, $M^\perp \subseteq M_1 \cap \cdots \cap M_n$ contrary to $M^\perp \neq \{0\}$. Therefore, $\mathfrak{M} = \{M_1, \cdots, M_n\}$.

THEOREM 1. *For an f-ring A the following statements are equivalent:*

(1) *A has the ascending chain condition for closed l-ideals.*

(2) *A has the descending chain condition for closed l-ideals.*

(3) *A is isomorphic to a subdirect sum of a finite number of totally ordered rings.*

PROOF. The equivalence of (1) and (2) is clear since the mapping $I \rightarrow I^\perp$ is a dual automorphism of the lattice of closed l-ideals of A. The implication (3)$\Rightarrow$(1) is trivial. Finally, the implication (1)$\Rightarrow$(3) follows immediately from Lemmas 1 and 4.

If A is an f-ring, then the set $N(A)$ of all nilpotent elements of A is an l-ideal called the *l-radical* of A [1]. Clearly, $N(A/N(A)) = \{0\}$ and $N(I) = N(A) \cap I$ for any l-ideal I of A. The l-radical $N(A)$ of A can also be characterized [5] as the intersection of all prime l-ideals of A. Recall [4] that in an f-ring A an l-ideal P is prime if and only if for all $a, b \in A$, $ab \in P$ implies $a \in P$ or $b \in P$. This is also equivalent to the property: A/P is totally ordered with no nonzero divisors of zero.

If A is an f-ring with $N(A) = \{0\}$, then it follows that if $S \subseteq A$ is nonvoid, its left annihilator, right annihilator, and $S^\perp$ coincide [1, p. 63].

LEMMA 5. *Let A be an f-ring with $N(A) = \{0\}$ and let P be an l-ideal of A. Then P is a complemented prime l-ideal if and only if P is a maximal closed l-ideal.*

Proof. If P is a complemented prime l-ideal, then $P^{\perp}\neq 0$, whence $P^{\perp}\not\subseteq P$. Since $P^{\perp}P^{\perp\perp}=\{0\}$, we have that $P^{\perp\perp}\subseteq P$, so that P is closed. Also, since P is prime, A/P is totally ordered. Therefore, by Lemma 1 (with $I=P^{\perp}$), we have that $P=P^{\perp\perp}$ is a maximal closed l-ideal.

Conversely, it will suffice to show that if P is maximal closed, then it is prime. But in this case $P^{\perp}$ is totally ordered by Lemma 1, so that if $a, b\notin P$, there exists a $c\in P^{\perp}$ such that $|a|\wedge|c|\neq 0$ and $|b|\wedge|c|\neq 0$. Now $N(P^{\perp})=N(A)\cap P^{\perp}=\{0\}$, so that $P^{\perp}$ is a prime f-ring. Therefore, since $|a|\wedge|c|$, $|b|\wedge|c|\in P^{\perp}$, we have

$$0\neq(|a|\wedge|c|)(|b|\wedge|c|)\leqq|a|\,|b|=|ab|.$$

Since $P^{\perp}\cap P=\{0\}$, it follows that $(|a|\wedge|c|)(|b|\wedge|c|)\notin P$. Thus $ab\notin P$, and P is prime.

Now from Lemma 4, Lemma 5, and Theorem 1 we readily conclude

Theorem 2. *Let A be an f-ring with $N(A)=\{0\}$. Then the following statements are equivalent:*

(1) *A has the ascending chain condition for closed l-ideals.*

(2) *A has the descending chain condition for closed l-ideals.*

(3) *A is isomorphic to a subdirect sum of a finite number of totally ordered rings having no nonzero divisors of zero.*

As we have now seen the ascending and descending chain conditions for *closed* l-ideals are equivalent in any f-ring. However, even for f-rings with zero l-radical the ascending and descending chain conditions for l-ideals need not be equivalent. For example, the f-ring $Q[\lambda]$, which has zero l-radical, satisfies the ascending but not the descending chain condition for l-ideals. Note, however, that if $N(A)=\{0\}$ and if A satisfies the descending chain condition for l-ideals, then A is isomorphic to a finite direct sum of l-simple totally ordered rings [1, Theorem 17] and therefore satisfies the ascending chain condition for l-ideals.

We also observe that in Theorems 1 and 2 the "subdirect sum" of statement (3) cannot be strengthened to "direct sum." For let A be the sub f-ring of the direct sum of two copies of $Q[\lambda]$ defined by

$$A=\{(f,g);\,f,g\in Q[\lambda]\ \text{with}\ f(0)=g(0)\}$$

Then $N(A)=\{0\}$ and A has the ascending chain condition for closed l-ideals, but A cannot be isomorphic to a direct sum of totally ordered rings.

2. **Chain conditions in J-semi-simple f-rings.** An l-ideal P of an f-ring A is *l-primitive* if and only if A/P is an l-simple ordered ring

with identity. Thus an l-primitive l-ideal is prime. The *J-radical*, $J(A)$, of A is the intersection of all l-primitive l-ideals of A. Clearly $N(A) \subseteq J(A)$. If $J(A) = \{0\}$, then A is *J-semi-simple*.[4]

The example $Q[\lambda]$ shows that in f-rings with zero l-radical closed prime l-ideals need not be l-primitive. However, for J-semi-simple f-rings we have

LEMMA 6. *If P is a closed prime l-ideal of a J-semi-simple f-ring A, then P is an l-primitive l-ideal and*

$$A = P \oplus P^{\perp}.$$

PROOF. By Lemmas 1 and 5, $P^{\perp}$ is totally ordered and so, since $J(P^{\perp}) = J(A) \cap P^{\perp} = \{0\}$ [**4**, p. 188], $P^{\perp}$ is an l-primitive f-ring. Let $e \in P^{\perp}$ be the identity for $P^{\perp}$. Then, since P is the right ring annihilator of $P^{\perp}$,

$$a = (a - ea) + ea \in P + P^{\perp}$$

for all $a \in A$. Thus, $A = P + P^{\perp}$ and since $P \cap P^{\perp} = \{0\}$, this sum is direct. Therefore A/P is isomorphic to $P^{\perp}$ and P is an l-primitive l-ideal.

THEOREM 3. *Let A be a J-semi-simple f-ring satisfying the ascending chain condition for closed l-ideals. Then the set $\mathcal{P}$ of closed prime l-ideals of A coincides with the set of l-primitive l-ideals of A, and A is the direct sum of the l-ideals $P^{\perp}$ $(P \in \mathcal{P})$.*

PROOF. By Lemmas 4 and 5, $\mathcal{P}$ is finite. Let $\mathcal{P} = \{P_1, \cdots, P_n\}$. By Lemmas 2 and 6 the sum

$$P_1^{\perp} + \cdots + P_n^{\perp}$$

is direct and each $P_i^{\perp}$ is an l-primitive f-ring. If $e_i \in P_i^{\perp}$ is the identity of $P_i^{\perp}$, then for each $a \in A$

$$a - \sum_{i=1}^{n} e_i a \in P_1 \cap \cdots \cap P_n.$$

By Lemma 4 this implies that

$$a = \sum_{i=1}^{n} e_i a,$$

whence

$$A = P_1^{\perp} \oplus \cdots \oplus P_n^{\perp}.$$

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[4] The notions of l-primitivity and of the J-radical as well as the structure theory of J-semi-simple f-rings are due to Johnson [**4**].

To complete the proof it will suffice, in view of Lemma 6, to show that each l-primitive l-ideal P of A is one of the P_i ($i=1, \cdots, n$). But since P is proper $P_i^\perp \not\subseteq P$ for some i and so, since P is prime $P_i \subseteq P$. However, P_i is l-primitive, hence maximal [4, p. 187]; thus $P_i = P$.

THEOREM 4. *For a J-semisimple f-ring A the following statements are equivalent*:

(1) *A has the ascending chain condition for closed l-ideals.*

(2) *A has the descending chain condition for closed l-ideals.*

(3) *A has the ascending chain condition for l-ideals.*

(4) *A has the descending chain condition for l-ideals.*

(5) *A is isomorphic to the direct sum of a finite set of l-simple totally ordered rings with identity.*

PROOF. The implication (1)$\Rightarrow$(5) is by Theorem 3. Also (1)$\Leftrightarrow$(2) by Theorem 1. By Theorem II.5.8 of [4] we have (4)$\Leftrightarrow$(5). Since (5)$\Rightarrow$(3)$\Rightarrow$(1) are trivial, the proof is complete.

3. **Remarks.** If an f-ring A satisfies the ascending chain condition for l-ideals (closed l-ideals), then each l-ideal (closed l-ideal) of A is principal. For the l-ideal (closed l-ideal) generated by $\{a_1, \cdots, a_n\}$ is also generated by $|a_1| \vee \cdots \vee |a_n|$. Conversely, if each l-ideal of an f-ring A is principal, then A satisfies the ascending chain condition for l-ideals. Such a converse is not valid, however, for closed l-ideals. For example, in the f-ring of all real-valued functions on the integers every closed l-ideal is principal, but this f-ring clearly does not have the ascending chain condition for closed l-ideals.

If A is an arbitrary ring with the descending chain condition for right ideals, then A has the ascending chain condition for right ideals if and only if the additive group of A contains no p^∞ group (Fuchs [2]). Certainly the additive group of an f-ring has this property since this group must be torsion free. However, let A be the ring whose additive group is that of $Q[\lambda]$ and with multiplication defined by

$$\left(\sum_{i=0}^{m} a_i\lambda^i\right)\left(\sum_{j=0}^{n} b_j\lambda^j\right) = \sum_{i=0}^{m} (b_0a_i)\lambda^i + \sum_{j=1}^{n} (a_0b_j)\lambda^j.$$

Order A by

$$a_0 + a_1\lambda + \cdots + a_m\lambda^m > 0$$

in case $a_0 > 0$ or $a_0 = 0$ and $a_m > 0$. Then A is a commutative f-ring with identity which satisfies the descending but not the ascending chain condition for l-ideals.

Finally, it is known [4, p. 213] that in an f-ring with zero l-radical the descending chain condition on l-ideals and the descending chain condition on right l-ideals are equivalent. The corresponding statement for ascending chain conditions fails. For it can be shown that in an example due to Johnson [**4**, pp. 208–209] we have an f-ring with zero l-radical which satisfies the ascending chain condition for l-ideals but not the ascending chain condition for right l-ideals.[5]

References

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[5] In the f-ring of this example the principal right l-ideals generated by the elements xa, x^2a, $\cdots$ form a properly ascending chain.