AN INVERSION INTEGRAL

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For an integral transformation which involves a Chebyshev polynomial in the kernel an inversion integral is known [4]. A similar problem involving a Legendre polynomial has been solved [1]. By somewhat different methods we demonstrate an inversion integral for an integral transformation in which the kernel involves a Gegenbauer polynomial, $C_{n}^{2}(x)$, also known as ultraspherical polynomials, and thus includes the known cases for $k = 0, 1$ respectively, where we take the standardization $C_{n}^{0}(x) = \lim_{x \to 0} x^{-1} C_{n}^{k}(x)$. We write, for integers $k$ and $n$ with $0 < k < n$,

$$F_{n}^{k}(x) = 2^{(k-1)/2} \pi^{-1/2} \Gamma(k/2)n! \cdot [\Gamma(n + k)]^{-1/2} (x^2 - 1)^{(k-1)/2} \Gamma(k/2)n!,$$

$$G_{n}^{k}(x) = 2^{(k-1)/2} \pi^{-1/2} \Gamma(k/2)(n - k - 1)! \cdot [\Gamma(n - 1)]^{-1/2} (1 - x^2)^{(k-1)/2} \Gamma(k/2)n!.$$

and from this standardization for $k = 0$ we have

$$F_{n}^{0}(x) = (2/\pi)^{1/2} (x^2 - 1)^{-1/2} T_{n}(x),$$

$$G_{n}^{0}(x) = (2/\pi)^{1/2} (1 - x^2)^{-1/2} T_{n-1}(x).$$

If $f^{(k+1)}(x)$ is sectionally continuous for $0 < a \leq x \leq 1$ and $f^{(m)}(1) = 0$ for $0 \leq m \leq k$, then

$$\int_{a}^{1} F_{n}^{k}(t/x)g(t)dt = f(x)$$

has the solution

$$g(t) = \int_{a}^{1} G_{n}^{k}(t/y)y^{-n+k+1}(-y^{-1}d/dy)^{k+1}[y^{n+k-1}f(y)]dy$$

for $0 < a \leq t \leq 1$.

Since the integral is in the form of a convolution with respect to the Mellin transformation, formal application of this transformation, use of tables [3], and manipulation leads to the suggested form of the solution.

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The solution can be verified directly by first proving that for \( u > 1 \)

\[
J(u) = \int_{1/u}^{1} F_n^k(vu)G_n^k(v)dv = (2^k k!)^{-1}(u^2 - 1)^k u^{n-k-1}.
\]

This integral can be rewritten in a standard form of a convolution for the Mellin transformation \([5; or 3, 6.1(13)],\)

\[
\int_0^\infty F_n^k(vu)U(vu - 1)G_n^k(v)[1 - U(v - 1)]dv,
\]

where \( U(x) = 0, x < 0; = 1, x > 0. \) Thus we have

\[
\mathfrak{M}\{J(u); s\} = \mathfrak{M}\{F_n^k(u)U(u - 1); s\} \mathfrak{M}\{G_n^k(u)[1 - U(u - 1)]; 1 - s\}.
\]

From Rodrigues' formula \([2, 10.9(11)]\) (noting that with the standardization for \( k = 0 \) this reduces to \([3, 10.11(14)]\)) and the tables \([3, 6.2(32), 6.1(10)]\) and after some simplification

\[
\mathfrak{M}\{F_n^k(u)U(u - 1); s\} = (2^k k!)^{-1/2} \Gamma[1 - s + n + 1] \Gamma[1 - k - n] \Gamma[1 - s + 1]^{-1},
\]

\[
\mathfrak{M}\{G_n^k(u)[1 - U(u - 1)]; s\} = \pi^{1/2} \Gamma(s) \Gamma[(2 + s + n + 1)/2] \Gamma[(s + n + 1)/2]^{-1},
\]

so that

\[
\mathfrak{M}\{J(u); s\} = (2^k k!)^{-1} B[(1 - s + n + 1)/2, k + 1].
\]

But also from the tables \([3, 6.2(32), 6.1(10)]\)

\[
(2^k k!)^{-1}(u^2 - 1)^k u^{n-k-1} U(u - 1)
\]

has the same Mellin transform, hence the formula follows.

Consider the iterated integral

\[
I(x) = \int_x^1 F_n^k(t/x) \left( \int_t^1 G_n^k(t/y) y^{-n-k+1}(-y^{-1}d/dy)^{k+1}[y^{n-k-1}f(y)]dy \right) dt
\]

which is formed by direct substitution of the proposed value for \( g(t) \) into the integral equation. If the order of integration is changed

\[
I(x) = \int_x^1 y^{-n-k+1}(-y^{-1}d/dy)^{k+1}[y^{n-k-1}f(y)] \left( \int_x^y F_n^k(t/x)G_n^k(t/y)dt \right) dy.
\]

Thus if we let \( v = t/y, u = y/x \) the inner integral becomes
\[ yJ(u) = (2^k k!)^{-1}(y^2 - x^2)^k y^{n-k} x^{-n-k+1}, \]

and we can write

\[ I(x) = \frac{1}{(2^k k!)^{-1} x^{-n-k+1}} \int_0^1 \left( y^2 - x^2 \right)^k d\left\{ (-y^{-1} d/dy)^k [y^{n+k-1} f(y)] \right\}. \]

Successive integrations by parts and application of the conditions \( f^{(m)}(1) = 0 \) then yields \( I(x) = f(x) \).

References


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