(3) converges to $1/(1+a_1' \cdot v')$. Therefore by Theorem 1 of [4] the assumption that some neighborhood of v does not contain infinitely many even approximants of $f(a)$ is false. Hence the theorem is proved for the case considered here. The other case is similar.

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A CHARACTERIZATION OF QF-3 ALGEBRAS

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Let A be an associative algebra with a unit element 1 and

$$0 \to A \to X_1 \to X_2 \to \cdots \to X_n \qquad (n > 0) \tag{1}$$

an exact sequence of A-A-homomorphisms with A-A-projective modules X_p, $1 \leqq p \leqq n$. Recently concerning complete homology of algebras, Nakayama [4] has proposed to classify algebras according to how long an exact sequence (1) they have. In this paper we shall show that the first class in his classification is the class of QF-3 algebras (for definition see Thrall [5]), that is to say, A is QF-3 if and only if A has an exact sequence $0 \to A \to X$, where X is A-A-projective.

To begin with we state

Lemma 1. *A is a* QF-3 *algebra if and only if A has a faithful left A-module which is projective and injective.*

This lemma was already used in [2] and [3] and for the proof we shall refer to Theorems 3.1 and 3.2 of [1].

Now we shall prove

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THEOREM 2. *Let A be an algebra with unit element finite over a field K. A is* QF-3 *if and only if A-A-module A can be embedded into a projective A-A-module X.*

PROOF. *Necessity.* Assume that A have an exact sequence of A-A-homomorphisms $0 \to A \to X$, where X is A-A-projective. As was proved by Nakayama [4], there is a faithful left ideal L of A such that the dual $\mathrm{Hom}(L, K)$ is monomorphic to a free right A-module P. Denote N the radical of A and $l(N)$ the left annihilator of N in A. If $l(N)e_\lambda \neq 0$, where e_λ is a primitive idempotent of A,[1] then e_λ must be an element of L. Indeed, suppose $e_\lambda \notin L$, then $\sum_{\kappa\neq\lambda} Ae_\kappa + Ne_\lambda$ contains L because $\sum_{\kappa\neq\lambda} Ae_\kappa + Ne_\lambda$ is the largest left ideal which does not contain e_λ. On the other hand $l(N)e_\lambda(\sum_{\kappa\neq\lambda} Ae_\kappa + Ne_\lambda) = 0$. Hence L is not faithful. Thus, if we denote by Λ the set of all indices such that $l(N)e_\lambda \neq 0$, we obtain $\sum_{\lambda\in\Lambda} Ae \subseteq L \subseteq A$. Hence we have at once a direct sum decomposition of

$$L\colon\ L = \sum_{\lambda\in\Lambda} Ae_\lambda \oplus \left(L \cap A\left(1 - \sum_{\lambda\in\Lambda} e_\lambda\right)\right).$$

Since $\mathrm{Hom}(\sum_{\lambda\in\Lambda} Ae_\lambda, K)$ is injective, $\mathrm{Hom}(\sum_{\lambda\in\Lambda} Ae_\lambda, K)$ is isomorphic to a direct summand of P. Hence $\mathrm{Hom}(\sum_{\lambda\in\Lambda} Ae_\lambda, K)$ is projective. Thus $\sum_{\lambda\in\Lambda} Ae_\lambda$ is projective and injective. Since every simple right ideal is isomorphic to a submodule of $\mathrm{Hom}(\sum_{\lambda\in\Lambda} Ae_\lambda, K)$, $\sum_{\lambda\in\Lambda} Ae_\lambda$ is faithful. Therefore by Lemma 1 A is a QF-3 algebra.

Sufficiency. In the following we shall denote by A^* and N^* the inverse isomorphic algebra of A and its radical respectively. An A-A-module A may be regarded as a left module over the algebra $A \otimes_K A^*$, by setting $(a \otimes b^*)x = axb$, $a, b, x \in A$ and $b^* \in A^*$. Assume that A is a QF-3 algebra. Let $\sum_{\kappa\in\Lambda} Ae_\kappa$ and $\sum_{\sigma\in\Sigma} e_\sigma A$ be a unique minimal faithful left A-module and a unique minimal faithful right A-module respectively. Then $\sum_{\sigma\in\Sigma} A^*e_\sigma^*$ is a unique minimal faithful left A^*-module. If $r(N)e_\kappa \approx \bar{A}\bar{e}_{\pi(\kappa)}$ (resp. $e_\sigma l(N) \approx \bar{e}_{\rho(\sigma)}\bar{A}$), then $\pi(\kappa) \in \Sigma$ (resp. $\rho(\sigma) \in \Lambda$), where $\bar{A} = A/N$ and $r(N) = \{x \in A \mid Nx = 0\}$. If $\mu \notin \Sigma$ and $\eta \notin \Lambda$, $e_\mu r(N) = 0$ and $l(N)e_\eta = 0$. It is known that $A \otimes_K A^*$ is also QF-3 and $\sum_{\kappa\in\Lambda} Ae_\kappa \otimes \sum_{\sigma\in\Sigma} A^*e_\sigma^*$ is a unique minimal faithful left $A \otimes A^*$-module.[2]

Now we shall prove that the left $A \otimes A^*$-module A is monomorphic to a direct sum of finite number of copies of $\sum_{\kappa\in\Lambda,\sigma\in\Sigma} Ae_\kappa \otimes A^*e_\sigma^*$. For this aim it is enough to show that every simple left $A \otimes A^*$-submodule of A is isomorphic to a submodule of $\sum_{\kappa\in\Lambda,\sigma\in\Sigma} Ae_\kappa \otimes A^*e_\sigma^*$,

[1] By the category isomorphism we may assume that A is a core algebra.

[2] See Theorem 17.7 in [3].

because $\sum_{\kappa\in\Lambda,\sigma\in\Sigma} Ae_\kappa\otimes A^*e_\sigma^*$ is injective. Denote $\mathfrak{N}$ the radical of $A\otimes A^*$. Then $\mathfrak{N}\supseteq N\otimes A^*+A\otimes N^*$. Hence the socle of $A\otimes A^*$-module A is contained in $r(N)\cap l(N)$ and the socle of $\sum_{\kappa\in\Lambda,\sigma\in\Sigma} Ae_\kappa\otimes A^*e_\sigma^*$ is contained in $\sum_{\kappa\in\Lambda,\sigma\in\Sigma} r(N)e_\kappa\otimes r(N^*)e_\sigma^*$. Clearly

$$r(N)e_\kappa \otimes r(N^*)e_\sigma^* \approx \overline{A}\bar{e}_{\pi(\kappa)} \otimes \overline{A}^*\bar{e}^*_{\rho(\sigma)} \approx (A \otimes A^*)\bar{e}_{\pi(\kappa)} \otimes \bar{e}^*_{\rho(\sigma)},$$

where $\overline{A}^*=A^*/N^*$. Hence let

$$e_{\pi(\kappa)} \otimes e^*_{\rho(\sigma)} = g_1 + g_2 + \cdots + g_t$$

be a decomposition of $e_{\pi(\kappa)}\otimes e^*_{\rho(k)}$ into primitive idempotents of $A\otimes A^*$, then every simple submodule of $r(N)e_\kappa\otimes r(N^*)e_\sigma^*$ is isomorphic to a $(A\otimes A^*)g_i/\mathfrak{N}g_i$, $1\leqq i\leqq t$, because $\overline{A}\otimes\overline{A}^*$ is an almost symmetric algebra. Therefore if f is obtained from a decomposition of $e_\mu\otimes e_\eta^*$ into primitive idempotents: $e_\mu\otimes e_\eta^*=f+\cdots$, and if any simple submodule of $\sum_{\kappa\in\Lambda,\sigma\in\Sigma} r(N)e_\kappa\otimes r(N^*)e_\sigma^*$ is not isomorphic to $(A\otimes A^*)f/\mathfrak{N}f$, then $\mu\notin\Sigma$ or $\eta\notin\Lambda$ and hence $(e_\mu\otimes e_\eta^*)(l(N)\cap r(N))=0$. Consequently $f(l(N)\cap r(N))=0$. It follows that every simple composition factor of $l(N)\cap r(N)$ is isomorphic to a simple submodule of $\sum_{\kappa\in\Lambda,\sigma\in\Sigma} r(N)e_\kappa\otimes r(N^*)e_\sigma^*$. This completes the proof.

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