

We are now ready to derive a formula $N(c)$. Let q_i be any odd prime for which -7 is a quadratic residue, and n_j any for which -7 is a quadratic non-residue. Let c , any positive odd integer, be written in the form

$$c = 7^g \prod_{i,j} q_i^{e_i} n_j^{f_j}, \quad g \geq 0, i \geq 0, j \geq 0, e_i \geq 1, f_j \geq 1.$$

THEOREM 2. *Let $N(1) = 13, N(3) = 8$, and for c any odd integer greater than 3, let $N(c)$ be the least common multiple of all the factors $7^g, q_i - 1, n_j + 1, q_i^{e_i - 1}, n_j^{f_j - 1}$, then if $n > N(c), |a_n| \neq c$.*

By [1], $|a_n| > 1$ if $n > 13$. By Theorem 1, if $n > 8, |a_n| \neq 3$. By Lemmas 12, 13, 14, and 15, $a_{N(c)} \equiv 0 \pmod{c}$. Suppose $|a_n| = c, c > 3$. By Theorem 1, this is true for only one n . By Lemma 7, this n must be a factor of $N(c)$, therefore $n \leq N(c)$. Thus for all values of c , if $n > N(c)$, then $|a_n| \neq c$.

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TWO NEW REPRESENTATIONS OF THE PARTITION FUNCTION

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MacMahon [1] defined a two-rowed partition of the positive integer n as a representation of the form $n = \sum_{i=1}^r a_i + \sum_{j=1}^s b_j$ where the a_i and b_j are positive integers subject to the conditions $r \geq s, a_i \geq a_{i+1}, b_j \geq b_{j+1}, a_i \geq b_i$. Such partitions may be conveniently visualized by placing the summands on two rows, the a_i on the top row and the b_j on the bottom row, with each b_i immediately beneath a_i . Thus for $n = 3$ the partitions in question are (omitting + signs)

$$\begin{array}{cccc} 3, & 21, & 2, & 111, & 11. \\ & & 1 & & 1 \end{array}$$

In this note the following two theorems will be proved.

THEOREM 1. *The number of two-rowed partitions of n satisfying $a_i > a_{i+1}, b_j > b_{j+1}$ is $p(n)$, the ordinary partition function of n .*

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sets A, B of cardinalities s and $s + 1$ respectively. If $k = 2s$ is even, then ${}_k C_{[k/2]} = {}_{2s} C_s$ is the number of ways of separating c_1, \dots, c_k into two sets A, B of cardinality s (with the separation B, A being regarded as distinct from A, B). Consider now the full partition $\pi: n = c_1 + \dots + c_k + 2c_{k+1} + \dots + 2c_{k+l}$, with $k + 2l$ parts. Every separation of c_1, \dots, c_k into two sets A, B can be uniquely extended to a separation of all the parts into sets A', B' of *distinct* integers (by placing one c_j in A and the other in B for each $j = k + 1, \dots, k + l$). Hence the number $\Delta(n)$ of δ -partitions of n is the same as the number of partitions of the form

$$n = d_1 + \dots + d_h + e_1 + \dots + e_{h+1} \quad (d_1 > \dots > d_h, e_1 > \dots > e_{h+1})$$

or of the form

$$d_1 + \dots + d_h + e_1 + \dots + e_h \quad (d_1 > \dots > d_h, e_1 > \dots > e_h)$$

(here $d_1 + \dots + d_h + e_1 + \dots + e_h$ and $e_1 + \dots + e_h + d_1 + \dots + d_h$ are counted as distinct). It follows that if $F_h(\alpha)$ denotes the number of partitions of α into exactly h distinct parts, then the number $\Delta(n)$ of δ -partitions of n has the value

$$\Delta(n) = \sum_{h=0}^{\infty} \sum_{\alpha=0}^n \{ F_h(\alpha) F_h(n - \alpha) + F_h(\alpha) F_{h+1}(n - \alpha) \}.$$

The generating function of $F_h(\alpha)$ is

$$\sum_{\alpha=0}^{\infty} F_h(\alpha) x^\alpha = \frac{x^{(h^2+h)/2}}{(1-x) \dots (1-x^h)} \quad (|x| < 1)$$

(cf. [2, Theorem 346]), and therefore

$$\begin{aligned} & \sum_{n=0}^{\infty} \Delta(n) x^n \\ &= \sum_{h=0}^{\infty} \left\{ \frac{x^{h^2+h}}{(1-x)^2 \dots (1-x^h)^2} + \frac{x^{(h+1)^2}}{(1-x)^2 \dots (1-x^h)^2 (1-x^{h+1})} \right\} \\ &= \sum_{r=0}^{\infty} \left\{ \frac{x^{r^2+r}}{(1-x)^2 \dots (1-x^r)^2} + \frac{x^{r^2}(1-x^r)}{(1-x)^2 \dots (1-x^{r-1})^2 (1-x^r)^2} \right\} \\ &= \sum_{h=0}^{\infty} \frac{x^{h^2}}{(1-x)^2 \dots (1-x^h)^2}. \end{aligned}$$

By virtue of an identity of Durfee (cf. [2, Theorem 351]), the lat-

ter series is equal to $\sum_{n=0}^{\infty} p(n)x^n$. The identity theorem for power series now yields $\Delta(n) = p(n)$, and the proof is complete.

I am indebted to Professor T. S. Motzkin and Mr. C. Sudler for suggesting simplifications in my original proof of Theorem 1.

PROOF OF THEOREM 2. Letting $\Delta_0(n)$ denote the number of δ -partitions of n into odd parts, we have to show that

$$\sum_{n=0}^{\infty} \Delta_0(n)x^n = (1+x)f(x^2)$$

where $f(x) = \sum_{n=0}^{\infty} p(n)x^n$; indeed it then follows at once that $\Delta_0(n) = p(\lfloor n/2 \rfloor)$.

It is evident that the proof of Theorem 1 can be repeated, with the only difference being that all the parts c_1, \dots, c_l are required to be odd. We thus obtain the expression

$$\Delta_0(n) = \left\{ \sum_{h=0}^{\infty} \sum_{\alpha=0}^n G_h(\alpha)G_h(n-\alpha) + G_h(\alpha)G_{h+1}(n-\alpha) \right\},$$

where $G_h(\alpha)$ is the number of partitions of α into exactly h distinct odd parts. The generating function of $G_h(\alpha)$ is

$$\sum_{\alpha=0}^{\infty} G_h(\alpha)x^\alpha = \frac{x^{h^2}}{(1-x^2)(1-x^4)\dots(1-x^{2h})}$$

(cf. [2, p. 279]). Hence

$$\begin{aligned} & \sum_{n=0}^{\infty} \Delta_0(n)x^n \\ &= \sum_{h=0}^{\infty} \left\{ \frac{x^{2h^2}}{(1-x^2)^2 \dots (1-x^{2h})^2} + \frac{x^{h^2+(h+1)^2}}{(1-x^2)^2 \dots (1-x^{2h})^2(1-x^{2h+2})} \right\} \\ &= \sum_{h=0}^{\infty} \frac{x^{2h^2}}{(1-x^2)^2 \dots (1-x^{2h})^2} \\ & \quad + x \sum_{h=0}^{\infty} \frac{x^{2h(h+1)}}{(1-x^2)^2 \dots (1-x^{2h})^2(1-x^{2h+2})}. \end{aligned}$$

The first series is $f(x^2)$ by Durfee's identity. To show that the second term is $xf(x^2)$ we must prove that

$$(1) \quad \sum_{h=0}^{\infty} \frac{x^{h(h+1)}}{(1-x)^2 \dots (1-x^h)^2(1-x^{h+1})} = f(x).$$

This can be seen combinatorially in much the same way as Durfee's identity. If π is a partition of n , its Ferrars graph contains a maximal $h \times (h+1)$ rectangle R , and the remaining points of the graph constitute two tails. If the tail to the right of R has α points and the tail below R has β points, there are $P_h(\alpha)P_{h+1}(\beta)$ ways of adjoining the tails to R , where $P_k(m)$ is the number of partitions of m into at most k parts. Hence

$$p(n) = \sum_{h=0}^{\infty} \sum_{\alpha+\beta=n-h(h+1)} F_h(\alpha)F_{h+1}(\beta),$$

and passing to the generating functions we obtain (1). This completes the proof of Theorem 2.

CONCLUDING REMARKS. Chaundy [3] has dealt with the more general problem of determining the number of r -rowed partitions of n with parts decreasing by at least ρ along rows and at least κ along columns. Theorem 1 is the case $r=2$, $\rho=1$, $\kappa=0$. I am unable to see that Chaundy's argument is correct, owing to the failure of his expression (11) to satisfy the proper initial conditions. In any event, Theorem 1 falls under the case $\rho+\kappa=1$, in which Chaundy's formula becomes indeterminate.

Theorem 2 can easily be transformed into the statement that the number of plane partitions of n possessing xy -symmetry, and whose parts do not exceed 2, is $p(\lfloor n/2 \rfloor)$. In this form it appears as a "theorem" in [1, p. 269]. However, the "proof" rests on the results of §520 which, as the reader is warned in §521, have not yet been rigorously established.

Finally, Theorems 1 and 2 suggest the problem of finding directly a 1:1 correspondence between the δ -partitions and ordinary partitions of n , and between the odd δ -partitions of n and the ordinary partitions of $\lfloor n/2 \rfloor$. As in the case of the Rogers-Ramanujan identities, this has not yet been done, and appears to be difficult.

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