We are now ready to derive a formula $N(c)$. Let q_i be any odd prime for which -7 is a quadratic residue, and n_j any for which -7 is a quadratic non-residue. Let c, any positive odd integer, be written in the form

$$c = 7^g \prod_{i,j} q_i^{e_i} n_j^{f_j}, \qquad g \geqq 0,\ i \geqq 0,\ j \geqq 0,\ e_i \geqq 1,\ f_j \geqq 1.$$

THEOREM 2. *Let $N(1)=13$, $N(3)=8$, and for c any odd integer greater than 3, let $N(c)$ be the least common multiple of all the factors 7^g, q_i-1, n_j+1, $q_i^{e_i-1}$, $n_j^{f_j-1}$, then if $n>N(c)$, $|a_n|\neq c$.*

By [1], $|a_n|>1$ if $n>13$. By Theorem 1, if $n>8$ $|a_n|\neq 3$. By Lemmas 12, 13, 14, and 15, $a_{N(c)}\equiv 0 \pmod{c}$. Suppose $|a_n|=c$, $c>3$. By Theorem 1, this is true for only one n. By Lemma 7, this n must be a factor of $N(c)$, therefore $n\leqq N(c)$. Thus for all values of c, if $n>N(c)$, then $|a_n|\neq c$.

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TWO NEW REPRESENTATIONS OF THE PARTITION FUNCTION

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MacMahon [1] defined a two-rowed partition of the positive integer n as a representation of the form $n=\sum_{i=1}^{r} a_i+\sum_{j=1}^{s} b_j$ where the a_i and b_j are positive integers subject to the conditions $r\geqq s$, $a_i\geqq a_{i+1}$, $b_j\geqq b_{j+1}$, $a_i\geqq b_i$. Such partitions may be conveniently visualized by placing the summands on two rows, the a_i on the top row and the b_j on the bottom row, with each b_i immediately beneath a_i. Thus for $n=3$ the partitions in question are (omitting + signs)

$$\begin{array}{ccccc} 3, & 21, & 2, & 111, & 11. \\ & & 1 & & 1 \end{array}$$

In this note the following two theorems will be proved.

THEOREM 1. *The number of two-rowed partitions of n satisfying $a_i>a_{i+1}$, $b_j>b_{j+1}$ is $p(n)$, the ordinary partition function of n.*

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THEOREM 2. *The number of two-rowed partitions of n with a_i, b_j odd and also satisfying $a_i > a_{i+1}$, $b_j > b_{j+1}$ is $p([n/2])$.*

As an example of Theorem 1, take $n=5$, getting the partitions

$$\begin{array}{ccccccc} 5, & 41, & 4, & 32, & 3, & 31, & 21. \\ & & 1 & & 2 & 1 & 2 \end{array}$$

There are $p(5)=7$ of these. Again if $n=9$ the partitions of Theorem 2 are

$$\begin{array}{ccccc} 9, & 71, & 531, & 53, & 51, \\ & 1 & & 1, & 3 \end{array}$$

and there are $p(4)=5$ of these.

PROOF OF THEOREM 1. Two-rowed partitions of n with $a_i > a_{i+1}$, $b_i > b_{i+1}$ will be referred to henceforth as *δ-partitions*. If the summands of such a partition are all written on one line we obtain an ordinary partition of n in which no integer appears more than twice as a part.

Conversely, let π be an ordinary partition of n in which no integer appears more than twice as a part. Such a partition is of the form $\pi: n = c_1 + \cdots + c_k + 2c_{k+1} + \cdots + 2c_l$, where $c_1 > c_2 > \cdots > c_k$, $c_{k+1} > c_{k+2} > \cdots > c_l$, and where $c_i \neq c_j$ for $i \neq j$. Then the c_i can be distributed on 2 rows to form a δ-partition, perhaps in several ways. Since the parts $c_{k+1}, \cdots, c_l$ must clearly appear once on each row, the number of δ-partitions arising from π depends only on k. We will now show that it equals ${}_kC_{[k/2]}$, the maximum binomial coefficient of order k. For this purpose it is convenient to introduce a more general quantity $Q(k, m)$, the number of δ-partitions formed from $c_1, \cdots, c_k$ with the number of parts on the top row exceeding that of parts on the bottom row by at least m. We thus wish to show that $Q(k, 0) = {}_kC_{[k/2]}$. Now for $m > 0$, $Q(k, m)$ satisfies the recurrence

$$Q(k, m) = Q(k-1, m-1) + Q(k-1, m+1).$$

This can be seen by observing that if c_k is on the top row, the remaining c's can be distributed in $Q(k-1, m-1)$ ways, and if it is on the bottom, in $Q(k-1, m+1)$ ways. When $m=0$ the recurrence still holds if we make the convention that $Q(k, -1) = Q(k, 0)$. This recurrence enables us to determine $Q(k, m)$ from the initial conditions $Q(1, m) = 1$ when $m \leqq 1$, $Q(1, m) = 0$ when $m \geqq 2$. The general formula is $Q(k, m) = {}_kC_{[(k-m)/2]}$, which gives the desired result when $m=0$.

Next observe that when $k = 2s+1$ is odd, ${}_kC_{[k/2]} = {}_{2s+1}C_s$ is the number of ways in which the parts $c_1, \cdots, c_k$ can be separated into two sets A, B of cardinalities s and $s+1$ respectively. If $k=2s$ is even, then ${}_kC_{[k/2]}={}_{2s}C_s$ is the number of ways of separating $c_1, \cdots, c_k$ into two sets A, B of cardinality s (with the separation B, A being regarded as distinct from A, B). Consider now the full partition π: $n=c_1+\cdots+c_k+2c_{k+1}+\cdots+2c_{k+l}$, with $k+2l$ parts. Every separation of $c_1, \cdots, c_k$ into two sets A, B can be uniquely extended to a separation of all the parts into sets A', B' of *distinct* integers (by placing one c_j in A and the other in B for each $j=k+1, \cdots, k+l$). Hence the number $\Delta(n)$ of δ-partitions of n is the same as the number of partitions of the form

$$n = d_1 + \cdots + d_h + e_1 + \cdots + e_{h+1} \; (d_1 > \cdots > d_h, e_1 > \cdots > e_{h+1})$$

or of the form

$$d_1 + \cdots + d_h + e_1 + \cdots + e_h \; (d_1 > \cdots > d_h, e_1 > \cdots > e_h)$$

(here $d_1+\cdots+d_h+e_1+\cdots+e_h$ and $e_1+\cdots+e_h+d_1+\cdots+d_h$ are counted as distinct). It follows that if $F_h(\alpha)$ denotes the number of partitions of α into exactly h distinct parts, then the number $\Delta(n)$ of δ-partitions of n has the value

$$\Delta(n) = \sum_{h=0}^{\infty}\sum_{\alpha=0}^{n}\left\{F_h(\alpha)F_h(n-\alpha) + F_h(\alpha)F_{h+1}(n-\alpha)\right\}.$$

The generating function of $F_h(\alpha)$ is

$$\sum_{\alpha=0}^{\infty} F_h(\alpha)x^\alpha = \frac{x^{(h^2+h)/2}}{(1-x)\cdots(1-x^h)} \quad (|x| < 1)$$

(cf. [**2**, Theorem 346]), and therefore

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Delta(n)x^n \\
&= \sum_{h=0}^{\infty}\left\{\frac{x^{h^2+h}}{(1-x)^2\cdots(1-x^h)^2} + \frac{x^{(h+1)^2}}{(1-x)^2\cdots(1-x^h)^2(1-x^{h+1})}\right\} \\
&= \sum_{r=0}^{\infty}\left\{\frac{x^{r^2+r}}{(1-x)^2\cdots(1-x^r)^2} + \frac{x^{r^2}(1-x^r)}{(1-x)^2\cdots(1-x^{r-1})^2(1-x^r)^2}\right\} \\
&= \sum_{h=0}^{\infty}\frac{x^{h^2}}{(1-x)^2\cdots(1-x^h)^2}.
\end{aligned}$$

By virtue of an identity of Durfee (cf. [**2**, Theorem 351]), the latter series is equal to $\sum_{n=0}^{\infty} p(n)x^n$. The identity theorem for power series now yields $\Delta(n)=p(n)$, and the proof is complete.

I am indebted to Professor T. S. Motzkin and Mr. C. Sudler for suggesting simplifications in my original proof of Theorem 1.

PROOF OF THEOREM 2. Letting $\Delta_0(n)$ denote the number of δ-partitions of n into odd parts, we have to show that

$$\sum_{n=0}^{\infty} \Delta_0(n)x^n = (1+x)f(x^2)$$

where $f(x)=\sum_{n=0}^{\infty} p(n)x^n$; indeed it then follows at once that $\Delta_0(n) = p([n/2])$.

It is evident that the proof of Theorem 1 can be repeated, with the only difference being that all the parts $c_1, \cdots, c_l$ are required to be odd. We thus obtain the expression

$$\Delta_0(n) = \left\{ \sum_{h=0}^{\infty} \sum_{\alpha=0}^{n} G_h(\alpha)G_h(n-\alpha) + G_h(\alpha)G_{h+1}(n-\alpha) \right\},$$

where $G_h(\alpha)$ is the number of partitions of α into exactly h distinct odd parts. The generating function of $G_h(\alpha)$ is

$$\sum_{\alpha=0}^{\infty} G_h(\alpha)x^\alpha = \frac{x^{h^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2h})}$$

(cf. [2, p. 279]). Hence

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Delta_0(n)x^n \\
&= \sum_{h=0}^{\infty} \left\{ \frac{x^{2h^2}}{(1-x^2)^2\cdots(1-x^{2h})^2} + \frac{x^{h^2+(h+1)^2}}{(1-x^2)^2\cdots(1-x^{2h})^2(1-x^{2h+2})} \right\} \\
&= \sum_{h=0}^{\infty} \frac{x^{2h^2}}{(1-x^2)^2\cdots(1-x^{2h})^2} \\
&\quad + x\sum_{h=0}^{\infty} \frac{x^{2h(h+1)}}{(1-x^2)^2\cdots(1-x^{2h})^2(1-x^{2h+2})}.
\end{aligned}$$

The first series is $f(x^2)$ by Durfee's identity. To show that the second term is $xf(x^2)$ we must prove that

$$\sum_{h=0}^{\infty} \frac{x^{h(h+1)}}{(1-x)^2\cdots(1-x^h)^2(1-x^{h+1})} = f(x). \tag{1}$$

This can be seen combinatorially in much the same way as Durfee's identity. If π is a partition of n, its Ferrars graph contains a maximal $h\times(h+1)$ rectangle R, and the remaining points of the graph constitute two tails. If the tail to the right of R has α points and the tail below R has β points, there are $P_h(\alpha)P_{h+1}(\beta)$ ways of adjoining the tails to R, where $P_k(m)$ is the number of partitions of m into at most k parts. Hence

$$p(n) = \sum_{h=0}^{\infty} \sum_{\alpha+\beta=n-h(h+1)} F_h(\alpha)F_{h+1}(\beta),$$

and passing to the generating functions we obtain (1). This completes the proof of Theorem 2.

Concluding remarks. Chaundy [3] has dealt with the more general problem of determining the number of r-rowed partitions of n with parts decreasing by at least ρ along rows and at least κ along columns. Theorem 1 is the case $r=2$, $\rho=1$, $\kappa=0$. I am unable to see that Chaundy's argument is correct, owing to the failure of his expression (11) to satisfy the proper initial conditions. In any event, Theorem 1 falls under the case $\rho+\kappa=1$, in which Chaundy's formula becomes indeterminate.

Theorem 2 can easily be transformed into the statement that the number of plane partitions of n possessing xy-symmetry, and whose parts do not exceed 2, is $p([n/2])$. In this form it appears as a "theorem" in [1, p. 269]. However, the "proof" rests on the results of §520 which, as the reader is warned in §521, have not yet been rigorously established.

Finally, Theorems 1 and 2 suggest the problem of finding directly a 1:1 correspondence between the δ-partitions and ordinary partitions of n, and between the odd δ-partitions of n and the ordinary partitions of $[n/2]$. As in the case of the Rogers-Ramanujan identities, this has not yet been done, and appears to be difficult.

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