W is n-parallelisable. Now by the main theorem of [2], ∂W bounds a contractible manifold, and so represents the zero element of Γ_{2n}.

References

1. C. T. C. Wall, *Classification of $(n-1)$-connected $2n$-manifolds*, Ann. of Math. **75** (1962), 163–189.

2. ———, *Killing the middle homotopy group of odd dimensional manifolds*, Trans. Amer. Math. Soc. **103** (1962), 421–433.

Trinity College, Cambridge

THE COEFFICIENTS IN THE EXPANSION OF CERTAIN PRODUCTS

L. CARLITZ[1]

1. The identities

$$\prod_{n=0}^{\infty}(1-p^nx)^{-1}=\sum_{n=0}^{\infty}\frac{x^n}{(1-p)(1-p^2)\cdots(1-p^n)},\tag{1}$$

$$\prod_{n=0}^{\infty}(1-p^nx)=\sum_{n=0}^{\infty}\frac{(-1)^np^{n(n-1)/2}x^n}{(1-p)(1-p^2)\cdots(1-p^n)},\tag{2}$$

where $|p|<1$, are well known. The more general products

$$\prod_{m,n=0}^{\infty}(1-p^mq^nx)^{-1},\qquad \prod_{m,n=0}^{\infty}(1-p^mq^nx)\qquad (|p|<1,\ |q|<1)$$

have been discussed in [1; 2].

In the present note we consider the products

$$\prod_{n=0}^{\infty}(1-p^nx-q^ny)^{-1},\ \prod_{n=0}^{\infty}(1-p^nx-p^ny)\ (|p|<1,\ |q|<1).\tag{3}$$

Put

$$F(x,y)=\prod_{n=0}^{\infty}(1-p^nx-q^ny)^{-1}=\sum_{r,s=0}^{\infty}A_{rs}x^ry^s,\tag{4}$$

where $A_{rs}=A_{rs}(p,q)$ is independent of x and y. It follows from (4) that

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$$(1 - x - y)F(x, y) = F(px, qy),$$

so that

$$(1 - p^r q^s)A_{rs} = A_{r-1,s} + A_{r,s-1} \qquad (r + s > 0). \tag{5}$$

Making use of (5) we get for the first few values of A_{rs}:

$$A_{00} = 1, \qquad A_{10} = \frac{1}{1 - p}, \qquad A_{01} = \frac{1}{1 - q},$$

$$A_{20} = \frac{1}{(1 - p)(1 - p^2)}, \qquad A_{02} = \frac{1}{(1 - q)(1 - q^2)},$$

$$A_{11} = \frac{1}{(1 - p)(1 - pq)} + \frac{1}{(1 - q)(1 - pq)}.$$

It is evident from (1) that

$$A_{r0} = \frac{1}{(1 - p)(1 - p^2) \cdots (1 - p^r)}, \qquad A_{0r} = \frac{1}{(1 - q)(1 - q^2) \cdots (1 - q^r)}. \tag{6}$$

Also it is clear from (3) that

$$A_{rs}(p, q) = A_{sr}(q, p). \tag{7}$$

In (5) take $s = 1$, so that

$$(1 - p^r q)A_{r1} = A_{r-1,1} + A_{r0}. \tag{8}$$

Making use of (6) and (8) we find that

$$A_{r1} = \sum_{j=0}^{r} \frac{1}{(1 - p) \cdots (1 - p^j)(1 - p^j q) \cdots (1 - p^r q)}. \tag{9}$$

We next take $s = 2$ in (5) and combine with (9) to get

$$A_{r2} = \sum_{0 \leq j \leq k \leq r} \frac{1}{(1-p) \cdots (1-p^j)(1-p^j q) \cdots (1-p^k q)(1-p^k q^2) \cdots (1-p^r q^2)}. \tag{10}$$

It is now not difficult to state the general result, namely

$$A_{rs} = \sum \frac{1}{(1-p) \cdots (1-p^{j_1})(1-p^{j_1} q) \cdots (1-p^{j_2} q) \cdots (1-p^{j_s} q^s) \cdots (1-p^r q^s)}, \tag{11}$$

where the summation is over all $j_1, j_2, \cdots, j_s$ such that

$$0 \leqq j_1 \leqq j_2 \leqq \cdots \leqq j_s \leqq r. \tag{12}$$

The proof of (11) is by induction on s and will be omitted.

As a partial verification of (11) we note that since the number of solutions of (12) is equal to

$$\binom{r+s}{r} = \frac{(r+s)!}{r!s!},$$

it follows that when $p=q$, (11) reduces to

$$A_{rs}(p, q) = \binom{r+s}{r} \frac{1}{(1-p)\cdots(1-p^{r+s})}$$

which agrees with (1).

2. Turning next to the second product in (3) we put

$$G(x, y) = \prod_{n=1}^{\infty} (1 - p^n x - q^n y) = \sum_{r,s=0}^{\infty} B_{rs} x^r y^s. \tag{13}$$

It follows from (13) that

$$(1 - px - qy)G(px, qy) = G(x, y),$$

so that

$$(1 - p^r q^s)B_{rs} = - p^r q^s (B_{r-1,s} + B_{r,s-1}). \tag{14}$$

Also comparing (13) with (2) we have

$$B_{r0} = \frac{(-1)^r p^{r(r+1)/2}}{(1-p)\cdots(1-p^r)}. \tag{15}$$

Now put

$$\overset{*}{B}_{rs} = \overset{*}{B}_{rs}(p, q) = B_{rs}\left(\frac{1}{p}, \frac{1}{q}\right).$$

Then (14) becomes

$$(1 - p^r q^s)\overset{*}{B}_{rs} = \overset{*}{B}_{r-1,s} + \overset{*}{B}_{r,s-1}. \tag{16}$$

Since by (15)

$$\overset{*}{B}_{r0} = \frac{1}{(1-p)\cdots(1-p^r)} = A_{r0},$$

comparison of (16) with (5) yields

$$\overset{*}{B}_{rs} = A_{rs}. \tag{17}$$

Therefore we have

$$B_{rs}(p, q) = A_{rs}\left(\frac{1}{p}, \frac{1}{q}\right) \tag{18}$$

and B_{rs} is determined explicitly by means of (11).

If we let $A_{rs}(j_1, \cdots, j_s)$ denote the summand in the right member of (11) and $A_{rs}^*(j_1, \cdots, j_s)$ the corresponding function with p, q replaced by p^{-1}, q^{-1}, respectively, it follows that

$$\begin{aligned} A_{rs}^*(j_1, \cdots, j_s) = (-1)^{r+s}&p^{r(r+1)/2+j_1+\cdots+j_s} \\ &q^{s(s+1)/2+rs-j_1-\cdots-j_s} \cdot A_{rs}(j_1, \cdots, j_s). \end{aligned} \tag{19}$$

Note that when $p=q$ the sum of the exponents on p and q is equal to

$$\frac{1}{2}r(r+1) + \frac{1}{2}s(s+1) + rs = \frac{1}{2}(r+s)(r+s+1),$$

which is correct.

In terms of $A_{rs}(j_1, \cdots, j_s)$ and $A_{rs}^*(j_1, \cdots, j_s)$ we have

$$A_{rs} = \sum A_{rs}(j_1, \cdots, j_s), \tag{20}$$

$$B_{rs} = \sum A_{rs}^*(j_1, \cdots, j_s), \tag{21}$$

where in each case the summation is over all $j_1, \cdots, j_s$ satisfying (12).

From the definition of $A_{rs}(j_1, \cdots, j_s)$ we have

$$A_{rs}(j_1, \cdots, j_s) = \frac{A_{j_s, s-1}(j_1, \cdots, j_{s-1})}{(1 - p^{j_s}q^s) \cdots (1 - p^r q^s)};$$

therefore (20) yields

$$A_{rs} = \sum_{j=0}^{r} \frac{A_{j,s-1}}{(1 - p^j q^s) \cdots (1 - p^r q^s)}, \tag{22}$$

which can also be obtained from (5).

We remark that the pair of formulas

$$(r+1)A_{r+1,s} = \sum_{j=0}^{r}\sum_{k=0}^{s}\binom{j+k}{j}\frac{A_{r-j,s-k}}{1 - p^{j+1}q^k}, \tag{23}$$

$$(s+1)A_{r,s+1} = \sum_{j=0}^{r}\sum_{k=0}^{s}\binom{j+k}{j}\frac{A_{r-j,s-k}}{1 - p^j q^{k+1}} \tag{24}$$

can be proved by logarithmic differentiation of (4).

3. We shall now determine the coefficients in the expansion

$$\prod_{n=0}^{\infty}(1 - p_1x_1 - p_2y - p_3z)^{-1} = \sum_{r,s,t=0} A_{rst}x^ry^sz^t. \tag{25}$$

A *triple* T is an ordered set of three non-negative integers i, j, k. Each of the triples $(i-1, j, k)$, $(i, j-1, k)$, $(i, j, k-1)$ precedes i, j, k; notation $T_1 \prec T$. A *chain* C is a set of triples:

$$T_1 \prec T_2 \prec \cdots \prec T_k,$$

where $T_1 = (1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$; T_k is the last element of C. Corresponding to the triple i, j, k is the factor $1 - p_1^ip_2^jp_3^k$; we put

$$\pi(C) = \prod_{T\in C}(1 - p_1^ip_2^jp_3^k), \tag{26}$$

where $1 - p_1^ip_2^jp_3^k$ corresponds to the triple $T = (i, j, k)$.

We shall show that

$$A_{rst} = \sum_C \frac{1}{\pi(C)}, \tag{27}$$

where the summation is over all chains with last element (r, s, t).

In the first place it is clear from (25) that

$$(1 - p_1^rp_2^sp_3^t)A_{rst} = A_{r-1,s,t} + A_{r,s-1,t} + A_{r,s,t-1} \tag{28}$$

and

$$A_{rs0} = A_{rs}(p_1, p_2), \quad A_{r-t} = A_{rt}(p_1, p_3), \quad A_{0st} = A_{st}(p_2, p_3), \tag{29}$$

where $A_{rs}(p_1, p_2)$ is defined by (4). Moreover A_{rst} is uniquely determined by means of (28) and (29).

Now if A_{rst} is defined by (27) it follows from (11) and (27) that (28) is satisfied. To show that (28) is also satisfied we remark that if C is a chain with last element (r, s, t), then deleting this element we are left with a chain whose last element is $(r-1, s, t)$, $(r, s-1, t)$ or $(r, s, t-1)$ and conversely. In view of (26), (28) follows immediately.

If we put

$$\prod_{n=1}^{\infty}(1 - p_1x - p_2y - p_3z) = \sum_{r,s,t=0}^{\infty} B_{rst}x^ry^sz^t \tag{30}$$

then, exactly as in the proof of (18), we have

$$B_{rst} = A_{rst}\left(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}\right). \tag{31}$$

It is clear from (26), (27) and (31) how the coefficients in the expansion of

$$\prod_{n=0}^{\infty} (1 - p_1x_1 - \cdots - p_kx_k)^{-1}$$

and

$$\prod_{n=1}^{\infty} (1 - p_1x_1 + \cdots + p_kx_k)$$

can be determined for all $k \geqq 1$.

Added in proof. Professor B. M. Bennett has kindly informed the writer that he has obtained a formula equivalent to (11) above in his paper: *On a rank-order test for the equality of probabilities in multinomial trials*. Also it is evident from his paper that the coefficients $A_{rs}(p, q)$ are of some statistical interest.

References

1. L. Carlitz, *The expansion of certain products*, Proc. Amer. Math. Soc. **7** (1956), 558–564.

2. E. M. Wright, *Partitions of multi-partite numbers*, Proc. Amer. Math. Soc. **7** (1956), 880–890.

Duke University