THE RADIUS OF GYRATION OF A CONVEX BODY[1]

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The purpose of this note is to establish an inequality of isoperimetric type for convex bodies. Let K be a bounded convex body and l a line through its centroid; let $\delta(K, l)$ be the supremum of distances of points of K from l, $I(K, l)$ the moment of inertia of K about l, and $g(K, l)$ the radius of gyration $[I(K, l)/\text{mass } K]^{1/2}$ of K about l. If K were not restricted to be convex, it is easy to show that the values of the ratio $g(K, l)/\delta(K, l)$ consist of all numbers in the open interval $(0, 1)$. However, under the restriction to convex K we shall show that the infimum of this ratio is $1/15^{1/2}$. The usual methods of the calculus of variations are unavailable, as is usual in problems concerning convex bodies.

If K is a bounded convex body in three-space, and l is a line in three-space, and $\delta(x, y, z; l)$ the distance from the point (x, y, z) to l, we define the mass, $m(K)$, centroid $(\bar{x}(K), \bar{y}(K), \bar{z}(K))$, moment of inertia $I(K, l)$ and radius of gyration $g(K, l)$ as usual:

$$\text{(1)} \quad \begin{aligned} m(K) &= \int_K dx\, dy\, dz, \qquad \bar{x}(K) = \int_K x\, dx\, dy\, dz/m(K), \text{ etc.,} \\ I(K, l) &= \int_K \delta(x, y, z; l)^2 dx\, dy\, dz, \qquad g(K, l) = [I(K, l)/m(K)]^{1/2}; \end{aligned}$$

and we define

$$\text{(2)} \qquad \delta(K, l) = \sup\{\delta(x, y, z; l) : (x, y, z) \in K\}.$$

It is obvious that none of these quantities change if we replace K by its closure. Likewise, if a linear mass-distribution in an interval $[a, b]$ is defined by a density $A(x)$ $(a \leqq x \leqq b)$, and c is a real number, we define

$$\text{(3)} \quad \begin{aligned} m(A) &= \int_a^b A(x)dx, \qquad \bar{x}(A) = \int_a^b xA(x)dx/m(A), \\ I(A, c) &= \int_a^b (x - c)^2 A(x)dx, \qquad g(A, c) = [I(A, c)/m(A)]^{1/2}, \\ \delta(A, c) &= \text{larger of } |a - c|,\ |b - c|. \end{aligned}$$

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We simplify the notation when $c=\bar{x}(A)$, writing

$$(4) \quad I(A) = I(A, \bar{x}(A)), \quad g(A) = g(A, \bar{x}(A)), \quad \delta(A) = \delta(A, \bar{x}(A)).$$

THEOREM. *If K is a bounded convex body and l a line through its centroid, and $\delta(K, l)$ is the supremum of distances of points of K from l and $g(K, l)$ the radius of gyration of K about l, then*

$$(5) \qquad g(K, l)/\delta(K, l) > 1/15^{1/2}.$$

In this statement the constant $1/15^{1/2}$ cannot be replaced by any larger number.

By the remark after (2), we may and shall restrict our attention to closed convex bodies. Given any K and l as in the theorem, we choose rectangular axes so that l is the z-axis and the positive x-axis contains a point at distance $\delta(K, l)$ from l. The projection of K on the x-axis is an interval $[a, b]$ with $-\delta(K, l) \leqq a < b = \delta(K, l)$. For each ξ in $[a, b]$ let $A(\xi)$ be the area of the intersection $K(\xi)$ of K with the plane $x=\xi$. By the Brunn-Minkowski theorem [**1**, p. 88] $[A(x)]^{1/2}$ $(a \leqq x \leqq b)$ is a concave function, obviously continuous in the open interval (a, b) and by the closure of K continuous at a and b also. Also, $\bar{x}(A) = \bar{x}(K) = 0$, so $\delta(A) = b = \delta(K, l)$. For the moments of inertia we have

$$\begin{aligned} I(K, l) &= \int_a^b \left\{ \int_{K(x)} (x^2 + y^2) dy\, dz \right\} dx \\ &> \int_a^b \int_{K(x)} x^2\, dy\, dz\, dx \\ &= \int_a^b x^2 A(x) dx = I(A), \end{aligned}$$

so

$$(6) \qquad g(A)/\delta(A) < g(K, l)/\delta(K, l).$$

Let $\mathfrak{F}$ be the family of functions A each defined, continuous and non-negative on some closed interval and having $[A]^{1/2}$ concave on that interval. The function A of the preceding paragraph belongs to $\mathfrak{F}$. So if we define

$$(7) \quad \begin{aligned} \mu &= \text{infimum of } g(K, l)/\delta(K, l) \text{ for all convex bodies } K \text{ and all} \\ &\qquad \text{lines through the centroid of } K, \\ \mu' &= \text{infimum of } g(A)/\delta(A) \text{ for all } A \text{ in } \mathfrak{F}, \end{aligned}$$

by (6) we have

(8) $$\mu' \leqq \mu.$$

On the other hand, let $A(x)(a \leqq x \leqq b)$ belong to $\mathfrak{F}$. For each positive ϵ let K_ϵ be the solid of revolution obtained by revolving the set $\{(x, y): a \leqq x \leqq b,\ 0 \leqq y \leqq \epsilon[A(x)/\pi]^{1/2}\}$ about the x-axis. Then K_ϵ is convex, and its centroid is $(\bar{x}(A), 0, 0)$. Let l be the line $x = \bar{x}(A)$, $y = 0$. It is easy to see that as ϵ tends to 0, $\delta(K_\epsilon, l)$ tends to $\delta(A)$. Also,

$$I(K_\epsilon, l) = \int_a^b [(x - \bar{x}(A))^2 + \epsilon^2 A(x)/4\pi]\epsilon^2 A(x)dx,$$

$$m(K_\epsilon) = \int_a^b \epsilon^2 A(x)dx,$$

so

$$\lim_{\epsilon \to 0} g(K_\epsilon, l) = \left[\int_a^b (x - \bar{x}(A))^2 A(x)dx \Big/ \int_a^b A(x)dx\right]^{1/2} = g(A).$$

Therefore as ϵ tends to 0, $g(K_\epsilon, l)/\delta(K_\epsilon, l)$ tends to $g(A)/\delta(A)$, and μ cannot be greater than μ'. This and (8) prove that

(9) $$\mu = \mu',$$

and (9) and (6) prove that for all K and l as in the theorem, $g(K, l)/\delta(K, l) > \mu$. It remains to prove that $\mu = 1/15^{1/2}$. The function $A(x) = (1-x)^2$ $(0 \leqq x \leqq 1)$ belongs to $\mathfrak{F}$, and we readily compute that for it

$$g(A)/\delta(A) = 1/\sqrt{15}.$$

Hence

(10) $$\mu \leqq 1/\sqrt{15}.$$

Suppose now that $\mu < 1/15^{1/2}$. Let $A_1, A_2, \cdots$ be a sequence of functions belonging to $\mathfrak{F}$ such that

(11) $$\lim_{n \to \infty} g(A_n)/\delta(A_n) = \mu.$$

The ratio $g(A)/\delta(A)$ is unaltered by transformations $x \to hx + k$ with $h > 0$, so we may suppose that all the A_n have $[0, 1]$ as domain. Also, the ratio is unaffected by multiplication of A by a positive constant, so we may suppose that all A_n have supremum 1. Then for each positive integer k all the functions A_n satisfy the Lipschitz condition with constant $1/k$ on the interval $[1/k, 1-1/k]$, so we may select a subsequence uniformly convergent on that interval. By the diagonal process we obtain a subsequence converging to a limit $A(x)$ for each x in $(0, 1)$. Then $[A]^{1/2}$ is concave, so $A(x)$ has limits as $x \to 0+$ and as $x \to 1-$. We assign $A(0)$ and $A(1)$ these respective values. Then A belongs to $\mathfrak{F}$ and satisfies

$$g(A)/\delta(A) = \mu. \tag{12}$$

Since multiplication by a positive constant leaves A in $\mathfrak{F}$ and leaves the left member of (12) unchanged, we may assume $m(A)=1$. Likewise, by applying the transformation $x \to 1-x$ if necessary, we may obtain $\bar{x}(A) \leqq 1/2$. From A we construct K_ϵ by rotation of $\epsilon[A/\pi]^{1/2}$, as before; this is convex and its centroid is $(\bar{x}(A), 0, 0)$, so [1, p. 52]

$$1/4 \leqq \bar{x}(A) \leqq 1/2. \tag{13}$$

To simplify notation we shall define

$$c = \bar{x}(A). \tag{14}$$

Let ϵ be any positive number less than 1; the O and o notation will be used for estimates with ϵ near 0. Define

$$\alpha = \int_0^\epsilon A(x)dx;$$

then $\alpha = 0(\epsilon)$. Let A_ϵ be the restriction of A to the subinterval $[\epsilon, 1]$. Then

$$m(A_\epsilon) = 1 - \alpha,$$

$$m(A_\epsilon)\bar{x}(A_\epsilon) = \int_\epsilon^1 xA(x)dx = c + \alpha O(\epsilon),$$

whence

$$\bar{x}(A_\epsilon) = c(1 + \alpha) + \alpha O(\epsilon).$$

Also

$$\begin{aligned} I(A_\epsilon) &= I(A_\epsilon, c) - [\bar{x}(A_\epsilon) - c]^2 m(A_\epsilon) \\ &= I(A) - \int_0^\epsilon (c - x)^2 A(x)dx + O(\alpha^2) \\ &= I(A) - c^2\alpha[1 + O(\epsilon)], \end{aligned}$$

whence

$$g(A_\epsilon)^2 = I(A)\{1 + \alpha[1 - c^2/I(A) + O(\epsilon)]\}.$$

Since

$$\delta(A_\epsilon) \geqq 1 - \bar{x}(A_\epsilon) = \delta(A) - c\alpha + \alpha O(\epsilon),$$

we have

$$[g(A_\epsilon)/\delta(A_\epsilon)]^2 \tag{15}$$
$$\leqq I(A)/\delta(A)^2\{1 + \alpha[1 - c^2/I(A) + 2c/\delta(A) + O(\epsilon)]\}.$$

By definition of A, with (12),

$$I(A)/\delta(A)^2 = [g(A)/\delta(A)]^2 = \mu^2 < 1/15, \tag{16}$$

so, (allowing the next equation to define C)

$$\begin{aligned} C &= 1 - c^2/I(A) + 2c/\delta(A) = 1 - c^2/\mu^2\delta(A)^2 + 2c/\delta(A) \\ &= -\left(\frac{c}{\mu\delta(A)} - \mu\right)^2 + \mu^2 + 1. \end{aligned} \tag{17}$$

Since $c/\delta(A) = c/(1-c)$, which is increasing on $[0, 1)$, and $1/4 \leqq c \leqq 1/2$, we have $c/\delta(A) \geqq 1/3$. Since $\mu^2 < 1/15$, we also have $c/\mu\delta(A) - \mu > 0$, and the right member of (17) is not decreased if we replace $c/\delta(A)$ by 1/3:

$$C \leqq 1 + \mu^2 - \left(\frac{1}{3\mu} - \mu\right)^2. \tag{18}$$

The derivative with respect to t of the function $\phi(t) = (1/(3t) - t)^2$ is

$$\phi'(t) = -\frac{2}{t}\left(\frac{1}{9t^2} - t^2\right),$$

which is negative for t in $(0, 1/3)$. Hence the right member of (18) is increased if we replace μ by the larger number $1/15^{1/2}$, so that

$$C < 1 + 1/15 - (15^{1/2}/3 - 1/15^{1/2})^2 = 0.$$

The coefficient $C - O(\epsilon)$ of α in (15) is therefore negative for all positive ϵ near 0, and α is positive, so for all such ϵ we have by (15) and (16)

$$g(A_\epsilon)/\delta(A_\epsilon) < \mu.$$

But this contradicts the definition of μ, and the proof is complete.

If K is a right circular cylinder with axis l, it is easily computed that $g(K, l)/\delta(K, l) = 1/2^{1/2}$. I conjecture, but have not been able to complete a proof, that $1/2^{1/2}$ is in fact the supremum of the ratio $g(K, l)/\delta(K, l)$ for all convex bodies K and all lines l through the centroid of K.

Bibliography

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