ON THE STIEFEL-WHITNEY CLASSES OF A MANIFOLD. II

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1. **Introduction.** This note is a complement to a previous paper of the same title [1]. In that paper it was proved that certain Stiefel-Whitney classes or dual Stiefel-Whitney classes modulo 2 of a differentiable manifold always vanished. In the present note, analogous theorems are proved about the *integral* Stiefel-Whitney classes of a manifold.

The following convention regarding notation will be followed consistently. The mod 2 Stiefel-Whitney classes are denoted by lowercase letters, w_i, while the integral Stiefel-Whitney classes are denoted by capital letters, W_i; the subscript denotes the dimension. A bar over the appropriate symbol denotes the *dual* Stiefel-Whitney class, integral or mod 2, thus: $\overline{W}_i$, $\bar{w}_i$. Of course the integral classes are only defined in odd dimensions.

Let M^n denote a compact, connected, orientable, differentiable n-manifold. We will prove the following three theorems about its integral Stiefel-Whitney classes:

THEOREM 1. *If n is even,* $\overline{W}_{n-1}=0$.

THEOREM 2. *If n is even,* $W_{n-1}=0$.

THEOREM 3. *If* $n \equiv 3 \bmod 4$, $W_{n-2}=0$.

Note that Theorem 1 implies Corollary 1 to Theorem I of [1] in the orientable case. Similarly, Theorem 2 implies Theorem II of [1], and Theorem 3 represents a strengthening of part of the conclusion of Theorem III of [1].

The main interest in Theorem 1 stems from the fact that A. Haefliger and M. Hirsch have recently proved [3] that any (compact, orientable, differentiable) M^n, $n>4$, is differentiably imbeddable in R^{2n-1} if and only if $\overline{W}_{n-1}=0$ for n even and $\bar{w}_{n-1}=0$ for n odd. Thus it follows from Theorem 1 above and Corollary 1 to Theorem I of [1] that such an M^n is always differentiably imbeddable in R^{2n-1} for $n>4$.

An interesting application of Theorem 2 is to 8-dimensional manifolds. According to [2, p. 170], for any compact, orientable M^8, $W_5=0$; by Theorem 2, $W_7=0$. Thus W_3 is the only integral Stiefel-

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Whitney class of an 8-dimensional manifold which can be nonzero. It is known that W_3 is the first obstruction to the existence of an almost complex structure on M^8, and if $W_3=0$, then W_7 is the second obstruction. Therefore if we remove a single point from any compact, orientable M^8 for which $W_3=0$, the resulting noncompact manifold admits an almost-complex structure.

Note also that Theorem 2 above and Theorem III of [1] imply that the first obstruction to defining a field of tangent 2-frames on a compact, orientable n-manifold always vanishes provided $n \not\equiv 1 \mod 4$. This raises the question of determining the second obstruction to such a field.

2. **A lemma.** Assume that M^n is a compact, connected orientable n-manifold. Let T^q denote the torsion subgroup of $H^q(M^n, \mathbf{Z})$. Let p be a prime number, and

$$\text{(S)} \qquad H^q(M, \mathbf{Z}) \xrightarrow{p} H^q(M, \mathbf{Z}) \xrightarrow{r} H^q(M, \mathbf{Z}_p) \xrightarrow{\delta^*} H^{q+1}(M, \mathbf{Z})$$

be the exact sequence associated with the coefficient sequence

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}_p \to 0.$$

The cup product is a bilinear form

$$\cup : H^q(M^n, \mathbf{Z}_p) \times H^{n-q}(M^n, \mathbf{Z}_p) \to H^n(M^n, \mathbf{Z}_p) \approx \mathbf{Z}_p.$$

According to the Poincaré Duality Theorem, this bilinear form is nondegenerate.

LEMMA 1. *For any q, $r(T^{n-q})$ is the annihilator of $r[H^q(M, \mathbf{Z})]$.*

PROOF. It is clear that $r(T^{n-q})$ is contained in the annihilator of $r[H^q(M, \mathbf{Z})]$. We will complete the proof by showing that $r(T^q)$ has the same rank (as a vector space over $\mathbf{Z}_p$) as the annihilator of $r(H^q)$. For this purpose we introduce the following notation:

$b_i = i$th Betti no. of $M =$ rank of $H^i(M, \mathbf{Z})$.

$c_i =$ number of cyclic summands in the p-primary component of T^i (i.e., the p-primary component of T^i is the direct sum of c_i cyclic subgroups).

According to the Poincaré Duality Theorem, $b_i = b_{n-i}$ and $c_i = c_{n-i+1}$. Consideration of the exact sequence (S) shows that the rank of the vector space $H^i(M, \mathbf{Z}_p)$ is $b_i + c_i + c_{i+1}$ while the ranks of the subspaces $r(H^i)$ and $r(T^i)$ are $b_i + c_i$ and c_i, respectively. Therefore the rank of the annihilator of $r(H^q)$ is

$$(b_{n-q} + c_{n-q} + c_{n-q+1}) - (b_q + c_q)$$
$$= (b_q + c_{n-q} + c_q) - (b_q + c_q) = c_{n-q}$$

which is precisely the rank of the subspace $r(T^{n-q})$, as was to be proved.

REMARK. This lemma is apparently well known; see [2, p. 169].

3. **Proof of Theorem 1.** We will use the exact sequence (S) and the preceding lemma for the case $p=2$. It is well known that

$$\overline{W}_{i+1} = \delta^*(\bar{w}_i), \ (i \text{ even});$$

in fact, this equation may be taken as the definition of $\overline{W}_{i+1}$. Hence by exactness of (S), to prove $\overline{W}_{n-1}=0$, it suffices to prove that $\bar{w}_{n-2} \in r(H^{n-2})$. By the preceding lemma, this is equivalent to proving that $\bar{w}_{n-2}$ annihilates the subspace $r(T^2) \subset H^2(M, \boldsymbol{Z}_2)$.

By Lemma 7 of [1], the homomorphism $H^2(M^n, \boldsymbol{Z}_2) \to H^n(M^n, \boldsymbol{Z}_2)$ defined by $x \to x \cdot \bar{w}_{n-2}$ is a sum of iterated Steenrod squares, which we may assume to be admissible on account of Adem's relations. By Lemma 4 of [1] we may assume that the excess of any such admissible iterated Steenrod square is 1 or 2. We will complete the proof by showing that for $x \in r(T^2)$,

$$Sq^I(x) = 0,$$

where Sq^I; $H^2(M^n, \boldsymbol{Z}_2) \to H^n(M^n, \boldsymbol{Z}_2)$ is any admissible iterated Steenrod square of excess 1 or 2 and degree $n-2$.

In case the excess is 1, then we must have $I=(2^j, 2^{j-1}, \cdots, 2, 1)$ for some integer $j \geqq 0$. But in this case it is clear that $Sq^I x=0$, for $x \in r(T^2)$, because

$$Sq^1 x = r\delta^*(x) = 0$$

by exactness of (S).

In case the excess is 2, by Lemma 5 of [1] there exists an admissible iterated Steenrod square, Sq^J, and a power of 2, $m=2^k$, such that

$$Sq^I x = (Sq^J x)^m$$

and J has excess 0 or 1. In case J has excess 0, then

$$Sq^I x = x^m$$

and it is obvious that $Sq^I x=0$ for $x \in r(T^2)$. In case J has excess 1, then $J=(2^j, 2^{j-1}, \cdots, 2, 1)$ for some integer $j \geqq 0$ exactly as before, and $Sq^J(x)=0$ for $x \in r(T^2)$ for the same reason as before. Thus in either case, $Sq^I(x)=0$ for $x \in r(T^2)$, as was to be proved.

4. **Proof of Theorem 2.** We will divide the proof into two cases, according as $n=4k+2$ or $n=4k$. In both cases, use will be made of the following lemma.

LEMMA 2. *For any integer i, U_i^2 is the reduction* mod 2 *of an integral cohomology class.*

PROOF. Here U_i denotes the class of Wu; the Stiefel-Whitney classes are defined in terms of the U_i by the formula

$$w_j = \sum_i Sq^{j-i}U_i.$$

It is readily seen by induction on j that the U_i may be expressed as polynomials in the Steenrod squares of the w_j, and hence as polynomials in the Stiefel-Whitney classes w_j (since any Steenrod square of a Stiefel-Whitney class may be expressed as a polynomial in the Stiefel-Whitney classes). Now it is well known that the square of any Stiefel-Whitney class, w_j^2, is the reduction mod 2 of an integral class; for j even, it is the reduction of a Pontrjagin class, while for j odd it is the reduction of W_j^2. Hence the square of any polynomial in the w_j is the reduction mod 2 of an integral class. In particular, U_i^2 is the reduction of an integral class.

First we will prove Theorem 2 for the case $n=4k+2$, the easiest case. In this case

$$w_{n-2} = w_{4k} = Sq^{2k}U_{2k} = U_{2k}^2$$

which is the reduction mod 2 of an integral class by the lemma. Hence

$$W_{n-1} = \delta^*(w_{n-2}) = 0$$

by the exactness of (S).

Next, we will prove Theorem 2 for the case $n=4k$. In this case

$$w_{n-2} = w_{4k-2} = Sq^{2k-2}U_{2k}.$$

To prove $W_{n-1}=0$, it suffices to prove that $x\cdot w_{n-2}=0$ for any $x\in r(T^2)$, as in the proof of Theorem 1. To achieve this, it obviously suffices to prove that for $m=2^q$, $q\geqq 0$, and $x\in r(T^2)$,

$$x^m Sq^{2k-2m}U_{2k} = x^{2m}Sq^{2k-4m}U_{2k}. \tag{1}$$

For, successive application of (1) with $m=1, 2, 4, 8, \cdots$ shows that

$$\begin{aligned} x\cdot w_{n-2} &= x\cdot Sq^{2k-2}U_{2k} = x^2\cdot Sq^{2k-4}U_{2k} \\ &= x^4\cdot Sq^{2k-8}U_{2k} = \cdots = 0. \end{aligned}$$

To prove (1), note first that since $Sq^1x=0$, the only nonzero Steenrod squares of x^m are

$$Sq^0x^m = x^m,$$
$$Sq^{2m}x^m = x^{2m}.$$

Therefore by Cartan's formula,

$$Sq^{2k-2m}(x^mU_{2k}) = x^mSq^{2k-2m}U_{2k} + x^{2m}Sq^{2k-4m}U_{2k}. \tag{2}$$

But by the definition of the U_i,

$$\begin{aligned} Sq^{2k-2m}(x^mU_{2k}) &= U_{2k-2m}(x^mU_{2k}) \\ &= (U_{2k-2m}x^m)U_{2k} = Sq^{2k}(U_{2k-2m}x^m) \\ &= (U_{2k-2m}x^m)^2 = U^2_{2k-2m}x^{2m}. \end{aligned} \tag{3}$$

By Lemma 2, U^2_{2k-2m} is the reduction mod 2 of an integral class; since $x\in r(T^2)$, $U^2_{2k-2m}x^{2m}\in r(T^n)$, i.e.,

$$U^2_{2k-2m}x^{2m} = 0. \tag{4}$$

Combining (2), (3), and (4) gives (1) as desired.

5. **Proof of Theorem 3.** Let $n=4k+3$; then

$$w_{n-3} = w_{4k} = Sq^{2k}U_{2k} = U^2_{2k}$$

which is the reduction mod 2 of an integral class by Lemma 2, and

$$W_{n-2} = \delta^*(w_{n-3}) = 0$$

by exactness of (S).

Bibliography

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