

GENERALIZED HADAMARD MATRICES

A. T. BUTSON

1. Introduction. A square matrix H of order h all of whose elements are p th roots of unity is called a *Hadamard matrix* ($H(p, h)$ matrix) if $HH^{CT} = hI$. It is known [4] that $H(2, h)$ matrices can exist only for values $h = 2$ and $h = 4t$, where t is a positive integer. Although it has been conjectured that $H(2, 4t)$ matrices exist for all positive integers t , their existence has been established [1; 3; 4; 5; 6; 7] for only the following values of h , where q denotes an odd prime:

$$(1.1) \quad h = 2^k;$$

$$(1.2) \quad h = q^k + 1 \equiv 0 \pmod{4};$$

$$(1.3) \quad h = h_1(q^k + 1) \text{ where } h_1 \geq 2 \text{ is the order of an } H(2, h) \text{ matrix};$$

$$(1.4) \quad h = h^*(h^* - 1) \text{ where } h^* \text{ is a product of numbers of forms (1.1) and (1.2);}$$

$$(1.5) \quad h = 172;$$

$$(1.6) \quad h = h^*(h^* + 3) \text{ where } h^* \text{ and } h^* + 4 \text{ both are products of numbers of forms (1.1) and (1.2);}$$

$$(1.7) \quad h = h_1 h_2 (q^k + 1) q^k \text{ where } h_1 \geq 2, h_2 \geq 2 \text{ are orders of } H(2, h) \text{ matrices;}$$

$$(1.8) \quad h = h_1 h_2 s (s + 3) \text{ where } h_1 \geq 2, h_2 \geq 2 \text{ are orders of } H(2, h) \text{ matrices and where } s \text{ and } s + 4 \text{ both are of the form } q^k + 1;$$

$$(1.9) \quad h = (r + 1)^2 \text{ where both } r \text{ and } r + 2 \text{ are prime or prime powers;}$$

$$(1.10) \quad h \text{ is a product of numbers of the forms (1.1)–(1.9).}$$

This list is taken from [2].

This paper is concerned with $H(p, h)$ matrices when $p > 2$. The main result is the construction of $H(p, 2^m p^k)$ matrices where p is a prime and $m \leq k$ are non-negative integers.

2. Elementary properties. Some easily established results concerning $H(p, h)$ matrices which will be used in the sequel are the following:

(2.1) The requirement that $HH^{CT} = hI$ is equivalent to the requirement that $H^{CT}H = hI$; i.e., the orthogonality of any two rows of H is equivalent to the orthogonality of any two columns of H .

(2.2) A permutation of the rows (columns) and multiplication of the elements of a row (column) by a fixed p th root of unity are elementary operations which leave invariant the Hadamard property.

(2.3) An $H(p, h)$ matrix can always be reduced to the standard form in which the initial row and column contain only the root 1.

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¹ It was noted by the referee that this result is known, and may be found in R. E. Bellman's *Introduction to matrix analysis*, McGraw-Hill, 1960, p. 27, problem 13.

(2.4) If $H = (h_{ij})$ is an $H(p, h)$ matrix in standard form, then

$$\sum_{j=1}^h h_{ij} = \sum_{j=1}^h h_{ij}^C = 0, \quad i = 2, 3, \dots, h;$$

$$\sum_{i=1}^h h_{ij} = \sum_{i=1}^h h_{ij}^C = 0, \quad j = 2, 3, \dots, h.$$

(2.5) If H_1 is an $H(p_1, h_1)$ matrix, H_2 is an $H(p_2, h_2)$ matrix, $h = h_1 h_2$, and $p = \text{l.c.m.}(p_1, p_2)$, then $H_1 \otimes H_2$ is an $H(p, h)$ matrix.

(2.6) If H_1 is an $H(p_1, h)$ matrix, γ is a primitive p_2 th root of unity, and $p = \text{l.c.m.}(p_1, p_2)$, then γH_1 is an $H(p, h)$ matrix.

3. Construction of $H(p, h)$ matrices. Throughout the remainder of this paper H will denote an $H(p, h)$ matrix in standard form and γ a fixed primitive p th root of unity.

When p is a prime, the requirement (2.4) that $\sum_{j=1}^h h_{2j} = 0$ can be written in the form $\sum_{j=0}^{p-1} k_j \gamma^j = 0$, where the k_j are non-negative integers satisfying $\sum_{j=0}^{p-1} k_j = h$. Using $1 = -\sum_{j=1}^{p-1} \gamma^j$, the condition becomes $\sum_{j=1}^{p-1} (k_j - k_0) \gamma^j = 0$, where $\sum_{j=0}^{p-1} k_j = h$. Since $\gamma, \gamma^2, \dots, \gamma^{p-1}$ are independent over the rational field, it is necessary that $k_j = k_0$ for $j = 1, 2, \dots, p - 1$. Hence $pk_0 = h$ and the following result has been established.

THEOREM 3.1. *When p is a prime, an $H(p, h)$ matrix can exist only for values $h = pt$, where t is a positive integer.*

The necessary condition that $h = 2$ or $h = 4t$ for $H(2, h)$ matrices has two obvious possible analogues for $H(p, h)$ matrices when p is a prime; namely, $h = p$ or $h = p^2 t$ and $h = p$ or $h = 2pt$. Results to follow in this section show that neither of these is necessary. The condition in the above theorem is the most stringent that has been obtained; and when p is not a prime, even this is not necessary as the following immediate consequence of (2.6) shows.

THEOREM 3.2. *It is possible to construct $H(2p, h)$ matrices for p arbitrary and h any value described in (1.1)–(1.10).*

By using (2.6) an $H(p, h)$ matrix can be constructed from an $H(p_1, h)$ matrix, where p_1 is a divisor of p . Such an $H(p, h)$ matrix can obviously be reduced by the elementary operations (2.2) to an $H(p_1, h)$ matrix; and, consequently, is considered as trivial.

Now let V be the matrix defined by $v_{ij} = \gamma^{ij}, i, j = 0, 1, \dots, p - 1$. Then $\sum_{j=0}^{p-1} v_{ij} v_{kj}^C = \sum_{j=0}^{p-1} \gamma^{(i-k)j}$. When $i = k$, then $\sum_{j=0}^{p-1} \gamma^{(i-k)j} = p$. Suppose $i \neq k$. If $(i - k, p) = 1$, then γ^{i-k} is a primitive p th root of

unity and $\sum_{j=0}^{p-1} \gamma^{(i-k)j} = 0$. If $(i-k, p) = d$ where $d > 1$, let $p = p_1 d$ and $i-k = i_1 d$. Then $(i-k)p_1 = i_1 d p_1 = i_1 p \equiv 0 \pmod{p}$, so that γ^{i-k} is a p_1 th root of unity. In this case $\sum_{j=0}^{p-1} \gamma^{(i-k)j} = d \sum_{j=0}^{p_1-1} \gamma^{(i-k)j} = 0$. This establishes the following theorem.

THEOREM 3.3. *The Vandermonde matrix V defined by $v_{ij} = \gamma^{ij}$, $i, j = 0, 1, \dots, p-1$, is a symmetric $H(p, p)$ matrix.¹*

If $p = p_1 p_2 \dots p_r$, where the p_j are distinct prime powers, γ_j is a primitive p_j th root of unity, and V_j the corresponding Vandermonde matrix, then permutation matrices P and Q exist such that $V = P(V_1 \otimes V_2 \otimes \dots \otimes V_r)Q$. However, V_j can not be so decomposed, so it would not have been sufficient to have proven the above theorem for p a prime.

Suppose p is odd, say $p = 2q + 1$, and let n be the smallest quadratic nonresidue of p . Denote by U that permutation matrix such that $W = VU$ has elements $w_{ij} = \gamma^{nij}$, $i, j = 0, 1, \dots, p-1$. Define the matrix Q by $q_{ij} = 0$ for $i \neq j$ and $q_{ii} = \gamma^{ai^2}$ for $i = 0, 1, \dots, p-1$. Then $C = QVQ$ and $B = Q^n W Q^n$ are, by (2.2), $H(p, p)$ matrices. Using $-2q \equiv 1 \pmod{p}$, it is easy to see that $c_{ij} = \gamma^{a(i-j)^2}$ and $b_{ij} = \gamma^{na(i-j)^2}$. Obviously now, $c_{ij} = c_{i+k, j+k}$ and $b_{ij} = b_{i+k, j+k}$ for $k = 0, 1, \dots, p-1$, so that C and B are cyclic matrices. Furthermore, C and B are symmetric matrices, and each contains at most $q+1$ distinct p th roots of unity.

Defining the product of two rows v_i and v_j of V to be that vector obtained by multiplying (mod p) the corresponding components of the two rows, it is noted that $v_i v_j = v_{i+j}$, so that the rows of V form a cyclic group with generator v_1 . Similarly, the columns of V , the rows of W , and the columns of W all form cyclic groups with generators v_1^T , $w_1 = v_n$, and $w_1^T = v_n^T$, respectively. From this observation it easily follows that $D^k V = V T^k$ and $D^{nk} W = W T^k$, where D is the matrix defined by $d_{ij} = 0$ for $i \neq j$, and $d_{ii} = \gamma^i$ for $i = 0, 1, \dots, p-1$, and T is the permutation matrix defined by $t_{i+1, i} = 1$ for $i = 0, 1, \dots, p-1$ and $t_{ij} = 0$ otherwise.

Let $Y = (11 \dots 1)$ and $Z = (00 \dots 0)$, both of length p . Then the k th column of B can be written in the form $T^k Q^n Y^T$, and the k th column of CP in the form $T^{kn} Q Y^T$. It will now be easy to prove the following construction theorem.

THEOREM 3.4. *When p is a prime, an $H(p, 2p)$ matrix can be constructed.*

The procedure will be to show that the matrix

$$K = \left(\begin{array}{c|c} QV & Q^nW \\ \hline (CP)^{CT} & B^{CT} \end{array} \right)$$

is an $H(p, 2p)$ matrix. First it is noted that $CY^T = (YQY^T)Y^T$ and $BY^T = (YQ^nY^T)Y^T$. When $p = 2q + 1$ is prime, there are q quadratic residues and q quadratic nonresidues of p . Consequently,

$$YQY^T + YQ^nY^T = \sum_{i=0}^{p-1} \gamma^{qi^2} + \sum_{i=0}^{p-1} \gamma^{nqi^2} = 2 \sum_{j=0}^{p-1} \gamma^{qj} = 0.$$

Thus $CY^T + BY^T = Z^T$. Now KK^{CT} in block form is

$$\left(\begin{array}{c|c} (QV)(QV)^{CT} + (Q^nW)(Q^nW)^{CT} & (QV)(CP) + (Q^nW)B \\ \hline (CP)^{CT}(QV)^{CT} + B^{CT}(Q^nW)^{CT} & (CP)^{CT}(CP) + B^{CT}B \end{array} \right).$$

By (2.2), QV , Q^nW , CP , and B are all $H(p, p)$ matrices. Thus $(QV)(QV)^{CT} + (Q^nW)(Q^nW)^{CT} = (CP)^{CT}(CP) + B^{CT}B = 2pI_p$. Now consider $(QV)(CP) + (Q^nW)B$. Using the fact that the k th columns of CP and B can be written as $T^{kn}QY^T$ and $T^kQ^nY^T$, respectively, the k th column of $(QV)(CP) + (Q^nW)B$ is then given by

$$\begin{aligned} QVT^{kn}QY^T + Q^nWT^kQ^nY^T &= D^{kn}QVQY^T + D^{kn}Q^nWQ^nY^T \\ &= D^{kn}(CY^T + BY^T) = D^{kn}Z^T = Z^T. \end{aligned}$$

Hence, $(QV)(CP) + (Q^nW)B$ and its conjugate transpose $(CP)^{CT}(QV)^{CT} + B^{CT}(Q^nW)^{CT}$ are both 0. Thus $KK^{CT} = 2pI_{2p}$ and the theorem is proven. An immediate consequence of this theorem and (2.5) is now stated.

THEOREM 3.5. *When p is a prime, $H(p, 2^m p^k)$ matrices can be constructed for any non-negative integers $m \leq k$.*

All the preceding results on the construction of $H(p, h)$ matrices are summarized in the following theorem.

THEOREM 3.6. *Let $p = 2^{k_0} p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the factorization of p into powers of distinct primes. If $k_0 = 0$, then $H(p, h_1)$ matrices can be constructed for $h_1 = 2^{j_0} p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r}$, where $j_i \geq 0$, $i = 0, 1, \dots, r$; $j_i > 0$ for at least one $i > 0$; and $j_0 \leq \sum_{i=1}^r j_i$. If $k_0 \neq 0$, then $H(p, h_1)$ matrices can be constructed for $h_1 = h_2 h_3$, where h_2 is 1 or the order of any $H(2, h)$ matrix, and h_3 is 1 or any value of h_1 .*

4. Remarks. Let the matrix obtained from H by deleting the initial row and column of 1's be called the core of H . Let π be a primitive root of the prime p . Then there exists a permutation matrix P such

that the core of PVP is the cyclic matrix whose rows are all the cyclic permutations of $(\gamma^{\pi}\gamma^{\pi^2} \cdots \gamma^{\pi^{p-1}})$. The rows of PVP obviously form a group. In a subsequent paper the connection between an $H(p, p^n)$ matrix whose rows form a group and whose core is cyclic, a maximal length linear recurring sequence with elements in $GF(p)$, and a "relative" difference set will be shown. One consequence of this connection is the following theorem.

THEOREM 4.1. *For any prime p and any positive integer n , an $H(p, p^n)$ matrix whose rows form a group and whose core is cyclic can be constructed.*

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