ON PRODUCTS OF STARLIKE FUNCTIONS

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Let $f(z)$ be regular in the unit disk, $|z|<1$, and normalized by $f(0)=0$, $f'(0)=1$. In accord with a definition of Robertson [4], the function $f(z)$ is said to be starlike at least of order α, where $0\leqq\alpha\leqq 1$, if $\operatorname{Re}\{zf'(z)/f(z)\}\geqq\alpha$ for $|z|<1$. Any such function is univalent and starlike in the unit disk [4; 7].

A role played by functions which are starlike of positive order in obtaining certain starlike products is given by the following result.

THEOREM 1. *Let $f_n(z)$, $n=1, 2, \cdots, N$, be starlike at least of order $1-d_n(\geqq 0)$, where $d_n\geqq 0$, and let $s_N=1-\sum_{n=1}^{N} d_n\geqq 0$. Then the product*

$$F_N(z) = z\prod_{n=1}^{N}\frac{f_n(z)}{z} \tag{1}$$

is starlike at least of order s_N. There exist functions of this type which are not starlike of any order greater than s_N.

PROOF. Clearly $F_N(z)$ is regular in the unit disk and $F_N(0)=0$, $F_N'(0)=1$. The logarithmic derivative of (1) and the hypothesis yield for $|z|<1$

$$\operatorname{Re}\frac{zF_{N'}(z)}{F_N(z)} = 1-N+\sum_{n=1}^{N}\operatorname{Re}\frac{zf_{n'}(z)}{f_n(z)} \geqq 1-N+\sum_{n=1}^{N}(1-d_n) = s_N. \tag{2}$$

Thus $F_N(z)$ is starlike at least of order s_N. If $f_n(z)\equiv z\exp(-d_nz)$, $n=1, 2, \cdots, N$, equality holds in (2) at $z=1$ and, therefore, the function $F_N(z)$ is not starlike of any order greater than s_N in this case.

For simplicity, Theorem 1 is stated in terms of finite products. The extension to infinite products is immediate if, when $N\to\infty$, s_N converges and $F_N(z)$ converges uniformly in any closed disk $|z|\leqq r<1$.

If the $f_n(z)$ in Theorem 1 are all identical to a starlike function $f(z)$ of order $1-d$, where $0\leqq d\leqq 1/N$, then $F_N(z)=z[f(z)/z]^N$ is starlike at least of order $1-Nd$. It is, however, not necessary to confine attention to integral powers when an appropriate branch of the power

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function is selected. The following extension of a result of Schild [**6**] shows this to be the case.

COROLLARY 1. *Let $f(z)$, where $f(0)=0, f'(0)=1$, be regular in $|z|<1$. For $a\geqq 1$, the function $F(z)=z[f(z)/z]^a$, $F'(0)=1$, is univalent and starlike in the unit disk if and only if $f(z)$ is starlike at least of order $1-1/a$.*

PROOF. The function

$$\operatorname{Re}\frac{zF'(z)}{F(z)} = 1 - a + a\operatorname{Re}\frac{zf'(z)}{f(z)}$$

is non-negative in $|z|<1$ if and only if $f(z)$ is starlike at least of order $1-1/a$.

The case where $0\leqq a<1$ is included by an application of the corollary to $f(z)=z[F(z)/z]^{1/a}$.

A regular univalent function is convex in the unit disk if it maps $|z|<1$ onto a convex region. A normalized convex function in the unit disk is known to be starlike at least of order $1/2$ [**3**; **8**]. In conjunction with Theorem 1, this proves

COROLLARY 2. *If $f_1(z)$ and $f_2(z)$, where $f_1(0)=f_2(0)=0$, $f_1'(0)=f_2'(0)=1$, are convex in $|z|<1$, then the function $f_1(z)f_2(z)/z$ is univalent and starlike in the unit disk. This product is not starlike of any order greater than zero when $f_1(z)\equiv f_2(z)\equiv z/(1+z)$.*

Since $f(z)$ is convex in $|z|<1$ if and only if $zf'(z)$ is starlike in $|z|<1$, Corollary 1 gives

COROLLARY 3. *Let $f(z)$, where $f(0)=0, f'(0)=1$, be regular in the unit disk. For $0<a\leqq 1$, the function $F(z)=z[f'(z)]^a$, $F'(0)=1$, is starlike at least of order $1-a$ if and only if $f(z)$ is convex in $|z|<1$.*

Since functions which are starlike of positive order are fundamental in the preceding results, a sufficient condition for a normalized regular function in $|z|<1$ to be of this type is given below. The result extends a theorem of Alexander [**1**] and a theorem of Schild [**6**]. The proof is based on a method used by Clunie and Keogh [**2**].

THEOREM 2. *The function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, $|z|<1$, is starlike at least of order α, where $0\leqq\alpha<1$, if*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqq 1-\alpha. \tag{3}$$

If $a_n\leqq 0$ for all n, then (3) is a necessary condition for $f(z)$ to be starlike at least of order α.

PROOF. Since by (3)

$$\sum_{n=2}^{\infty} |a_n| \leqq \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| \leqq 1,$$

in the unit disk $f(z) \neq 0$ for $z \neq 0$ and

$$\begin{aligned} &|zf'(z) - f(z)| - (1-\alpha)|f(z)| \\ &\quad = \left| \sum_{n=2}^{\infty} (n-1)a_n z^n \right| - (1-\alpha)\left| z + \sum_{n=2}^{\infty} a_n z^n \right| \\ &\quad \leqq \sum_{n=2}^{\infty} (n-1)|a_n| |z|^n - (1-\alpha)\left[|z| - \sum_{n=2}^{\infty} |a_n| |z|^n \right] \\ &\quad \leqq |z| \left[\sum_{n=2}^{\infty} (n-\alpha)|a_n| - (1-\alpha) \right] \\ &\quad \leqq 0. \end{aligned}$$

Therefore $|1 - zf'(z)/f(z)| \leqq 1-\alpha$, which implies $\operatorname{Re}\{zf'(z)/f(z)\} \geqq \alpha$ in $|z| < 1$.

In case $f(z)$ is starlike at least of order α and $a_n \leqq 0$ for all n, the function

$$f'(z) - \alpha\frac{f(z)}{z} = 1 - \alpha + \sum_{n=2}^{\infty} (n-\alpha)a_n z^{n-1} \neq 0$$

for any z in the unit disk. Moreover, in the interval $0 \leqq z < 1$ this function is continuous and, since it is positive at $z=0$, positive. It is therefore defined and non-negative at $z=1$. This proves the necessity of (3) for this class of functions.

For example, the function $f_n(z) = z - z^3/4n^2$ is by (4) starlike at least of order $\alpha_n = 1 - d_n = 1 - 2/(4n^2-1)$. Since $\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} 2/(4n^2-1) = 1$, Theorem 1 shows that

$$\frac{2}{\pi} \sin \frac{\pi}{2} z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{4n^2}\right)$$

is univalent and starlike in $|z| < 1$. This function is not univalent in $|z| < R$ for any $R > 1$ since its derivative has a zero at $z = 1$. These results were obtained by a different method in [5].

More generally, the Weierstrass Factor Theorem gives the following application of Theorem 1 to a class of entire functions.

THEOREM 3. *Let $f(z)$ be an entire function of the form*

$$f(z) = ze^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left\{\sum_{j=1}^{p_n} \frac{1}{j}\left(\frac{z}{a_n}\right)^j\right\}, \tag{4}$$

where $g(z)$, $g(0)=0$, is an entire function, $a_n \neq 0$ for all n, $p_n \geqq 0$ are certain integers, and where for each n in which $p_n=0$ the exponential factor does not appear. If the function $z\exp\{g(z)\}$ *is starlike at least of order α in $|z|<1$, if $|a_n|>1$ for all n, and if*

$$\sum_{n=1}^{\infty} |a_n|^{-p_n}[|a_n| - 1]^{-1} \leqq \alpha, \tag{5}$$

then $f(z)$ is univalent and starlike in $|z|<1$.

PROOF. Let $f_n(z)/z$ denote the $(n+2)$th factor of the infinite product (4). For $|z|<1$

$$\begin{aligned} \left|\frac{zf_n'(z)}{f_n(z)} - 1\right| &= \left| - \frac{z}{a_n - z} + \sum_{j=1}^{p_n}\left(\frac{z}{a_n}\right)^j\right| \\ &= \left|\frac{z^{p_n+1}}{a_n^{p_n}(a_n - z)}\right| \leqq \frac{|z|^{p_n+1}}{|a_n|^{p_n}(|a_n| - |z|)} \\ &\leqq |a_n|^{-p_n}(|a_n| - 1)^{-1} = d_n \leqq 1. \end{aligned}$$

Therefore $f_n(z)$ is starlike at least of order $1-d_n$ $(\geqq 0)$. Since the product (4) is uniformly convergent in the closed disk $|z|\leqq 1$, the function $f(z)$ is univalent and starlike in the unit disk by (5) and Theorem 1.

The fact that this theorem in some cases leads to sharp results is contained in the following corollary.

COROLLARY 4. *If the zeros a_n of an entire function $f(z)$ of the form* (4) *all lie on a ray* $\arg z=\phi$ *and if $g(z)\equiv(\alpha-1)e^{-i\phi}z$, where $0\leqq\alpha\leqq 1$, $f(z)$ is univalent and starlike for $|z|<r_0$, where r_0 is the modulus of the zero of $f'(z)$ which is closest to the origin and on the ray* $\arg z=\phi$.[3]

PROOF. A change of variable followed by an appropriate normalization shows that the assumption $r_0=1$ causes no loss of generality. The logarithmic derivative of $f(z)$ gives

$$z\frac{f'(z)}{f(z)} = 1 + (\alpha - 1)e^{-i\phi}z - \sum_{n=1}^{\infty} \frac{z^{p_n+1}}{a_n^{p_n}(a_n - z)}\cdot \tag{6}$$

[3] The special case of this corollary in which the infinite product represents an entire function whose zeros are of genus zero was proved independently by H. S. Wilf [9].

For $z=re^{i\phi}$, $0\leqq r<\min|a_n|$, the function (6) is a real valued continuous function of r. It is, moreover, positive at $r=0$ and obviously negative for r in the interval and sufficiently near $\min|a_n|$. Therefore $f'(z)$ has a zero at some $z_1=r_1e^{i\phi}$, where $0<r_1<\min|a_n|$. By hypothesis $f'(z)$ has no zero on the ray $\arg z=\phi$ and in the unit disk, and hence $|a_n|>1$ for all n. Finally, (6) is zero at $z=e^{i\phi}$. This proves that (5) holds for this case. The corollary now follows from Theorem 3.

By way of illustration, the function

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n},$$

where $\Gamma(z)$ is the Euler gamma-function, is univalent and starlike for $|z|<r_0$, where r_0 is the modulus of the largest negative zero $(=-.50\cdots)$ of $\Gamma'(z)$, and the result is sharp.

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